MECE 480 midterm solution 2017-2018 spring

Q1- (a)

$$G(z)K(z) = \frac{z+0.5}{(z-1.5)z}K = K\frac{N(z)}{D(z)} = K\frac{z+0.5}{z^2-1.5z} = K\frac{N(z)}{D(z)} \qquad \deg(N(z))=1 \quad \deg(D(z))=2$$

$$\left(\frac{d}{dz}N(z)\right)D(z) - \left[\frac{d}{dz}D(z)\right]N(z) = 0$$

5 point

$$1\left[z^2-1.5z\right] - \left[2z-1.5\right](z+\underbrace{0.5}_{a}) = 0 \rightarrow \text{put } z=0.5$$

$$z^2-1.5z-2z^2+1.5z-1z+0.75=0$$

$$-z^2-z+\frac{3}{4}=0 \Rightarrow z^2+z-\frac{3}{4}=0$$

$$\left(z+\frac{3}{2}\right)\left(z-\frac{1}{2}\right)=0$$

4 point

$z=-\frac{3}{2}$ $\quad z=\frac{1}{2}$ → potential break-in break-away points

(b) Put $z=\frac{1}{2}=0.5$ at equation $1+G(z)C(z)=0$

$$1+K\frac{0.5+0.5}{(0.5-1.5)0.5}=0 \qquad 1+K\frac{1}{-1\times 0.5}=0$$

1 point

$$1-2K=0 \qquad K=\frac{1}{2}>0 \text{ (in root locus)}$$

(b) Starting points $=1.5, 0$ (1) $\quad n=\deg$ of $(N(z))=1$ (0.5)

Ending points $=-0.5, \infty$ (1) $\quad m=\deg$ of $(D(z))=2$ (0.5) (1)

num_branches $=\max(m,n)=2$ (0.5)

Number of asymptotes: $|n-m|=1$ (1)

Angle of asymptotes $=180°$ (1)

$$\text{Centroid} = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{|n-m|} = \frac{(1.5)+(0)-(-0.5)}{|2-1|} = \frac{2}{1} = 2 \quad (1)$$

Break-in break away points

$z=-\frac{3}{2}$ $\quad z=\frac{1}{2}$ (2)

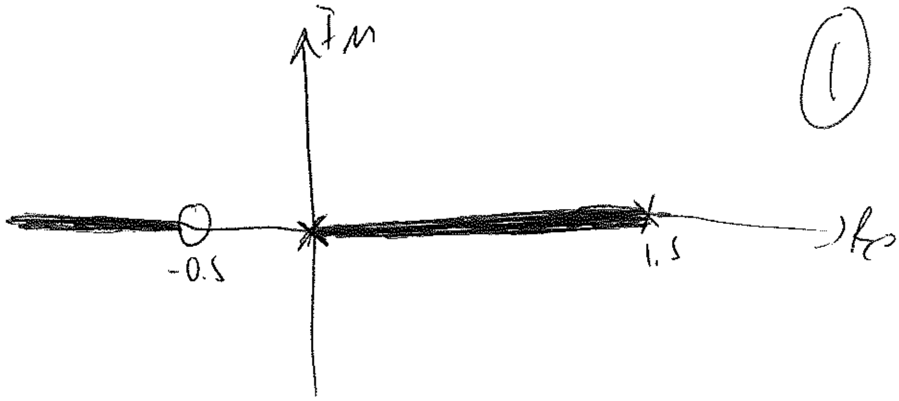
$$1+K\frac{-1.5+0.5}{(1.5)^2-1.5(-1.5)}=0 \qquad K=\frac{1}{2}>0$$

$$1+K\frac{-1}{4.5}=0$$

$K=4.5>0$ → both in root locus

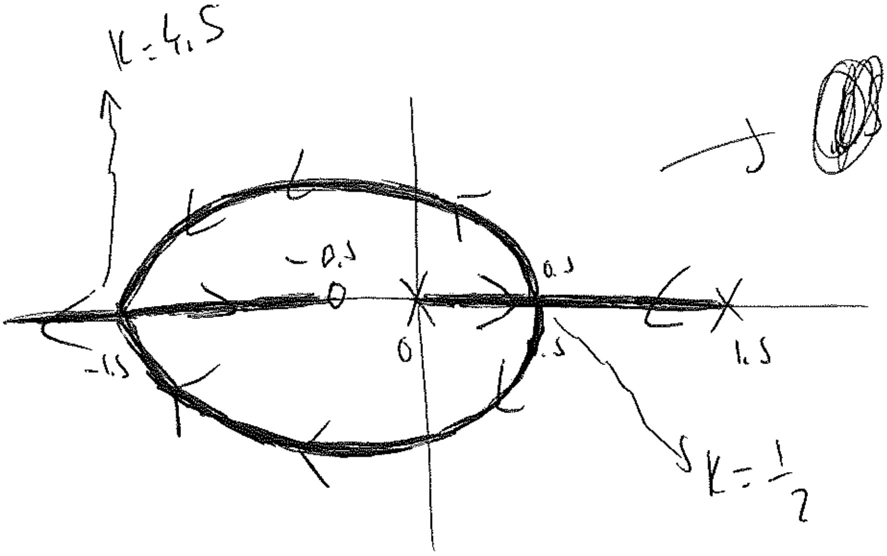
Q1 (b) continue

root locus on the real axis



1

total root locus plot



6

(c) Imaginary axis crossing $z=jw$

$$1+K\frac{z+0.5}{(z-1.5)\,z}=0 \qquad 1+K\frac{jw+0.5}{(jw-1.5)\,jw}=0$$ 3 point

$$(jw-1.5)\,jw+K(jw+0.5)=0$$

$$-w^2-1.5jw+Kjw+K\times 0.5=0$$

$$(K\times 0.5-w^2)+j(Kw-1.5w)=0$$

4 point

solution 1

$w=0 \quad K=0$ 1 point

solution 2

$K=1.5$ 2 point

$$1.5\times 0.5-w^2=0$$

$$\frac{3}{4}=w^2 \qquad w=\mp\frac{\sqrt{3}}{2}$$

Q1 (c) continue

if $K=0$ $w=0$ $z=j0$ $r=0$

if $K=1.5$ $w=i\frac{\sqrt{3}}{2}$ $z=0+j\frac{\sqrt{3}}{2}$ (imaginary axis crossing)

(d) Intersection with the unit circle $z=x+jy$

such that $x^2+y^2=1$ (a point on the unit circle in z-plane)

2 point

$$1+K\frac{z+a}{(z-1.5)z}=0 \qquad 1+K\frac{(x+yj+0.5)}{(x+yj-1.5)(x+yj)}=0$$

$$x^2-y^2+2xyj-1.5x-1.5yj+Kx+Kyj+K0.5=0$$ (seperate real and imaginary parts)

$$x^2-y^2-1.5x+Kx+K0.5=0$$

$$2xy-1.5y+Ky=0$$

$$x^2+y^2=1$$

(Q2) $G(s) = \dfrac{\ln 3 \quad s}{\left(s - \ln \frac{3}{2}\right)\left(s - \ln \frac{1}{2}\right)}$

$$G(z) = \frac{z-1}{z} \, Z\left\{ \mathcal{L}^{-1}\left\{ \frac{G(s)}{s} \right\}\Big|_{t=kT} \right\}$$

3 point

$$\frac{G(s)}{s} = \frac{\ln 3}{\left(s - \ln\frac{3}{2}\right)\left(s - \ln\frac{1}{2}\right)} = \frac{A}{s - \ln\frac{3}{2}} + \frac{B}{s - \ln\frac{1}{2}}$$

$$\ln 3 = A\left(s - \ln\frac{1}{2}\right) + B\left(s - \ln\frac{3}{2}\right)$$

$$\ln 3 = (A+B)s - \left[A\ln\frac{1}{2} + B\ln\frac{3}{2}\right]$$

$A + B = 0$ $\qquad \ln 3 = -\left[A\ln\frac{1}{2} + B\ln\frac{3}{2}\right]$

$B = -A$ $\qquad \ln 3 = -\left[A\ln\frac{1}{2} - A\ln\frac{3}{2}\right]$

$$\ln 3 = -\left[A\left(\ln\frac{1}{2} - \ln\frac{3}{2}\right)\right] = -A\left[\ln \frac{1}{2} \times \frac{2}{3}\right]$$

$\ln 3 = -A\ln\frac{1}{3}$ $\qquad \ln 3 = A\ln 3$ $\qquad A = 1$

$B = -1$

$$\frac{G(s)}{s} = \frac{A}{s - \ln\frac{3}{2}} + \frac{B}{s - \ln\frac{1}{2}} = \frac{1}{s - \ln\frac{3}{2}} - \frac{1}{s - \ln\frac{1}{2}}$$

6 point

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = \left(e^{\left[\ln\frac{3}{2}\right]t} - e^{\left(\ln\frac{1}{2}\right)t}\right)u(t) = \left(\left(\frac{3}{2}\right)^t - \left(\frac{1}{2}\right)^t\right)u(t)$$

2 point

$$\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\Big|_{t=kT} = \left[\left(\frac{3}{2}\right)^{kT} - \left(\frac{1}{2}\right)^{kT}\right]u(kT) \qquad u(k) = \sigma_k$$

$$T=1 \qquad = \left(\left(\frac{3}{2}\right)^{k} - \left(\frac{1}{2}\right)^{k}\right)u(k) = \left(\left(\frac{3}{2}\right)^{k} - \left(\frac{1}{2}\right)^{k}\right)\sigma_k$$

2point

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\Big|_{\substack{t=kT\\T=1}}\right\} = \left[\frac{1}{1-\frac{3}{2}z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right]$$

3point

$$= \frac{z}{z-\frac{3}{2}} - \frac{z}{z-\frac{1}{2}}$$

$$G(z) = \frac{z-1}{z}\,\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\Big|_{t=kT}\right\} = \frac{z-1}{z}\left[\frac{z}{z-\frac{3}{2}} - \frac{z}{z-\frac{1}{2}}\right]$$

$$G(z) = \frac{z-1}{z-\frac{3}{2}} - \frac{z-1}{z-\frac{1}{2}} = z-1\,\frac{\left[z-\frac{1}{2}-\left(z-\frac{3}{2}\right)\right]}{\left(z-\frac{3}{2}\right)\left(z-\frac{1}{2}\right)}$$

$$G(z) = \frac{(z-1)\,[1]}{\left(z-\frac{3}{2}\right)\left(z-\frac{1}{2}\right)} = \frac{z-1}{\left(z-\frac{3}{2}\right)\left(z-\frac{1}{2}\right)}$$

4 point

Q3

$$x_{k+1} = \overbrace{\begin{bmatrix} 0.5 & 0.25 \\ 1 & 0.5 \end{bmatrix}}^{A_d} x_k + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{b_d} u_k \qquad y_k = \overbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}^{c_d^T} u_k$$

(a) $$(zI - A_d) = \begin{bmatrix} z-0.5 & -0.25 \\ -1 & z-0.5 \end{bmatrix}$$ (2) point

$$(zI - A_d)^{-1} = \frac{\begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix}}{(z-0.5)^2 - (-1)(-0.25)}$$

$$(zI - A_d) = \frac{\begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix}}{z^2 - z + 0.25 - 0.25} = \frac{\begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix}}{z(z-1)}$$

3 point

$$G(z) = c_d^T (zI - A_d)^{-1} b_d$$

$$= \frac{\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{z(z-1)}$$

5 point

$$= \frac{\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} z-0.5+0.25 \\ z-0.5+1 \end{bmatrix}}{z(z-1)} = \frac{\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} z-0.25 \\ z+0.5 \end{bmatrix}}{z(z-1)}$$

$$= \frac{-z+0.25+z+0.5}{z(z-1)} = \frac{0.75}{z(z-1)}$$

(Q3)

(b) if $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ zero initial condition we will find zero-state solution then

$$Y(z) = G(z)\,U(z) \longrightarrow u_k = \sigma_k \rightarrow U(z) = \frac{z}{z-1}$$

$$Y(z) = \frac{0.75}{z(z-1)} \frac{z}{z-1} = z^{-1}\frac{0.75\,z}{(z-1)^2} \longrightarrow y_k = 0.75\,(1)^{(k-1)}(k-1)\,\sigma_{k-1}$$

$$= 0.75\,(1)^{k-1}(k-1)\,\sigma_{k-1}$$

8 puan

2 puan (unbounded result)

"(b)" (c) The system is not BIBO stable as bounded input in part results with unbounded output, Or $G(z) = \frac{0.75}{z(z-1)}$ poles of $G(z)$ $z=0$ $z=1$ ($z=1$ is over unit circle, not BIBO stable)

5 puan

(d) $u_k = 0$ 10 puan

Alternative eigenvalues $z=0$ $z=1$
not asym stable, but zero input stable

$$x_{k+1} = \begin{bmatrix} 0.5 & 0.25 \\ 1 & 0.5 \end{bmatrix} x_k$$

$$zX(z) - z x_0 = A_d X(z)$$

$$zX(z) - z\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0.5 & 0.25 \\ 1 & 0.5 \end{pmatrix} X(z)$$

$$\left[zI - A_d\right] X(z) = z\begin{pmatrix} a \\ b \end{pmatrix}$$

$$X(z) = (zI - A_d)^{-1} z \begin{pmatrix} a \\ b \end{pmatrix}$$

$$X(z) = \frac{z}{z(z-1)} \begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$zI - A_d = \begin{bmatrix} z-0.5 & -0.25 \\ -1 & z-0.5 \end{bmatrix}$$

$$(zI - A_d)^{-1} = \frac{\begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix}}{(z-0.5)^2 - 0.25}$$

$$(zI - A_d)^{-1} = \frac{\begin{bmatrix} z-0.5 & 0.25 \\ 1 & z-0.5 \end{bmatrix}}{z(z-1)}$$

$$X(z) = \frac{1}{z-1}\begin{bmatrix} az - a0.5 + b0.25 \\ bz - b0.5 + a \end{bmatrix}$$

$$X(z) = \begin{bmatrix} \frac{az - a0.5 + b0.25}{z-1} \\ \frac{bz - b0.5 + a}{z-1} \end{bmatrix} \rightarrow \text{due to } z-1$$

* Due to $z-1$ at denominator the system is not asymptotically stable but it is stable