MECE 480 midterm solutions

2018-2019 spring

Q1 a) $H(z) = K\,\dfrac{z+a}{z+b}\,\dfrac{1}{z-1} = K\,\dfrac{z+a}{z^2+(b-1)z-b} = \dfrac{N(z)}{D(z)}$

$$\left(\frac{dN(z)}{dz}\right)D(z) - \left(\frac{dD(z)}{dz}\right)N(z) = 0$$

$$1\left(z^2+(b-1)z-b\right) - \left[2z+b-1\right](z+a) = 0$$ (6 point)

put ~~z = -1+√2~~

$$\cancel{\left[(-1+\sqrt{2})^2+(b-1)(-1+\sqrt{2})-b\right] - \left[2(-1+\sqrt{2})+b-1\right]\left[(-1+\sqrt{2})+a\right] = 0}$$

$$\cancel{\left[3-2\sqrt{2}+(b-1)(-1+\sqrt{2})-b\right] - \left[-2+2\sqrt{2}+b-1\right]\left[(-1+\sqrt{2})+a\right] = 0}$$

$$\cancel{\left(3-2\sqrt{2}+1-\sqrt{2}+b(-2+\sqrt{2})\right] - \left(-3+2\sqrt{2}+b\right)\left(-1+\sqrt{2}+a\right) = 0}$$

$$\cancel{\left(4-3\sqrt{2}+b(-2+\sqrt{2})\right) -}$$

$$z^2+(b-1)z-b-2z^2-(b-1+2a)z-(b-1)a = 0$$

$$-z^2-2az-\left[b+(b-1)a\right] = 0$$

$$\boxed{z^2+2az+\left(b+(b-1)a\right] = 0}$$ (1 point)

put $z = -1+\sqrt{2}$

$$(-1+\sqrt{2})^2+2a(-1+\sqrt{2})+\left(b+(b-1)a\right] = 0$$ (0.5 point)

$$3-2\sqrt{2}+2a(-1+\sqrt{2})+\left(b+(b-1)a\right) = 0 \quad *$$

$z = -1-\sqrt{2}$

$$(-1-\sqrt{2})^2+2a(-1-\sqrt{2})+\left(b+(b-1)a\right) = 0$$ (0.5 point)

$$3+2\sqrt{2}+2a(-1-\sqrt{2})+\left(b+(b-1)a\right) = 0 \quad **$$

$* - **$: $-4\sqrt{2}+4\sqrt{2}\,a = 0$ $a=1$ (1 point)

$a=1$

$3-2\sqrt{2}+2a\,(-1+\sqrt{2})+(b+(b-1)a)=0$

$3-2\sqrt{2}+2\,(-1+\sqrt{2})+(b+(b-1))=0$

$1+(b+(b-1))=0$

$2b=0 \quad b=0$ 1 puan

$H(z)=K\dfrac{z+1}{z(z-1)} = K\dfrac{N(z)}{D(z)}$ $\deg(N(z))=1=m$, $\deg(D(z))=2=n$

(b) starting point: $0, 1$
ending points: $-1, \infty$
num-branches: $\max(n,m)=2$
centroid: $\dfrac{\sum_{i=1}^{n} \text{poles} - \sum_{i=1}^{m} \text{zeros}}{n-m} = \dfrac{(0+1)-(-1)}{(2-1)} = \dfrac{2}{1} = 1$
asymptotes: 180^0 3 puan

break-in break-away points

$1+H(z)=0 \qquad 1+K\dfrac{z+1}{z(z-1)}=0 \qquad z(z-1)+K(z+1)=0$

let $z=-1+\sqrt{2}$ (double pole at break-in point)

$z(z-1)+K(z-1)=0$

$(-1+\sqrt{2})(-1+\sqrt{2}-1)+K(-1+\sqrt{2}-1)=0$

$(-1+\sqrt{2})(-2+\sqrt{2})+K(-2+\sqrt{2})=0$

$(-1+\sqrt{2})+K=0 \qquad K=\sqrt{2}-1$

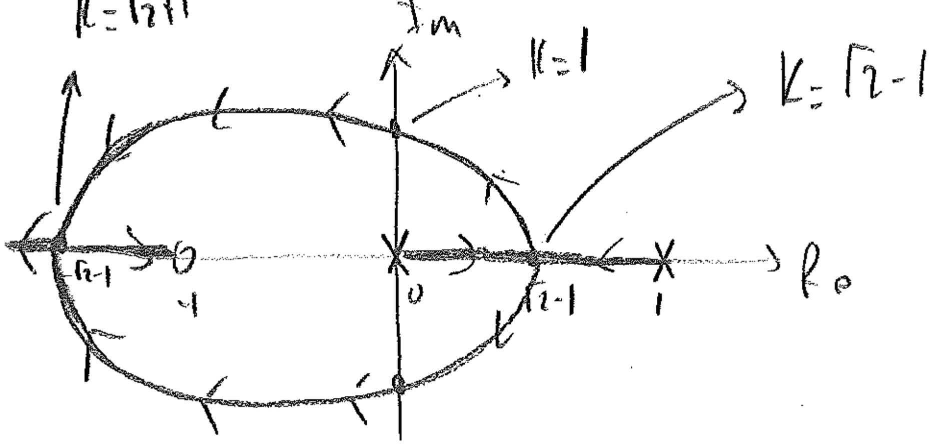
let $z=-1-\sqrt{2}$ (double pole at break in point)

$z(z-1)+K(z-1)=0$

$(-1-\sqrt{2})(-1-\sqrt{2}-1)+K(-1-\sqrt{2}-1)=0$

$K=\sqrt{2}+1$ 3 puan

root locus plot



5 puan

(c) $1+H(z) = 1+K\dfrac{z+1}{z(z-1)} = 0$ let $z = mj$

3 puan

$1+K\dfrac{mj+1}{mj(mj-1)} = 0$

~~$j^2 - j + K(j+1) = 0$~~

~~$(-1-j) + K(j+1) = 0$~~

~~$-(1+j) + K(1+j) = 0$~~

~~$K=1$~~

5 puan

$(K-m^2)+j(Km-m) = 0$

$K = m^2$
$Km = m$

$\boxed{m=0,\ K=0}$

or

$\boxed{K=1 \quad m=\mp 1}$ → the point where root locus passes imaginary axis

2 puan

(d) $1+H(z)=0$ let $z = a+bj$ $|z| = \sqrt{a^2+b^2} = 1$

3

$1+K\dfrac{a+bj+1}{(a+bj)(a+bj-1)} = 0$

8

$(a+bj)(a+bj-1) + K(a+bj+1) = 0$

$(a^2-b^2+2abj-a-bj) + K(a+1) + Kbj = 0 \Rightarrow a^2-b^2-a+K(a+1)+j(2ab-b+Kb)=0$

$a^2-b^2-a+K(a+1) = 0$ //
$2ab-b+Kb = 0$ //
$a^2+b^2 = 1$

→ 1 puan

(Q2)

$$\dot{x} = \underbrace{\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{b} u \qquad y = \underbrace{[1 \quad 0]}_{c^T} x$$

$$(sI - A) = \begin{bmatrix} s+1 & -1\\ 0 & s+1\end{bmatrix} \qquad (sI-A)^{-1} = \frac{\begin{bmatrix} s+1 & 1\\ 0 & s+1\end{bmatrix}}{(s+1)^2 - (0)(-1)} =$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2}\\ 0 & \frac{1}{s+1}\end{bmatrix} \Rightarrow e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$e^{At} = \begin{bmatrix} e^{-t} & te^{-t}\\ 0 & e^{-t}\end{bmatrix}$$

10 puan

$T = 0.01$ second

$$e^{AT} = \begin{bmatrix} e^{-0.01} & (0.01)\, e^{-0.01}\\ 0 & e^{-0.01}\end{bmatrix} = A_d$$

4 puan

$$b_d = \int_0^T e^{A\tau} b \, d\tau = \int_0^{0.01} \begin{bmatrix} e^{-\tau} & \tau e^{-\tau}\\ 0 & e^{-\tau}\end{bmatrix} \begin{bmatrix}0\\ 1\end{bmatrix} d\tau$$

$$= \int_0^{0.01} \begin{bmatrix} \tau e^{-\tau}\\ e^{-\tau}\end{bmatrix} d\tau = \begin{bmatrix} -(\tau e^{-\tau} + e^{-\tau})\\ -e^{-\tau}\end{bmatrix}\Bigg|_0^{0.01}$$

4 puan

$$b_d = \begin{bmatrix} -\left[(0.01 e^{-0.01} + e^{-0.01}) - (0 + e^{0})\right]\\ -\left[e^{-0.01} - e^{0}\right]\end{bmatrix} = \begin{bmatrix} 1 - 1.01\, e^{-0.01}\\ 1 - e^{-0.01}\end{bmatrix}$$

(Q2)

$c_d = c = [1 \quad 0]$ (1 puan)

Discretized system

$$x_{k+1} = A_d x_k + b_d u_k \qquad y_k = c_d^T x_k$$

$$x_{k+1} = \underbrace{\begin{bmatrix} e^{-0.01} & (0.01)e^{-0.01} \\ 0 & e^{-0.01} \end{bmatrix}}_{A_d} x_k + \underbrace{\begin{bmatrix} 1 - 1.01e^{-0.01} \\ 1 - e^{-0.01} \end{bmatrix}}_{b_d} u_k$$

(1 puan)

$$y_k = \underbrace{[1 \; 0]}_{c_d^T} x_k$$

(Q3) $x_{k+1} = \underbrace{\begin{bmatrix} 0.5 & 1 \\ 0.75 & -0.5 \end{bmatrix}}_{A_d} x_k + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b_d} u_k \qquad y_k = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C_d} x_k$

(a)

$$G(z) = c_d (zI - A_d)^{-1} b_d$$

$$(zI - A_d) = \begin{bmatrix} z-0.5 & -1 \\ -0.75 & z+0.5 \end{bmatrix}$$ 3 puan

$$(zI - A_d)^{-1} = \frac{\begin{bmatrix} z+0.5 & 1 \\ 0.75 & z-0.5 \end{bmatrix}}{(z-0.5)(z+0.5)-(-1)(-0.75)} = \frac{\begin{bmatrix} z+0.5 & 1 \\ 0.75 & z-0.5 \end{bmatrix}}{z^2-0.25-0.75} = \frac{\begin{bmatrix} z+0.5 & 1 \\ 0.75 & z-0.5 \end{bmatrix}}{z^2-1}$$

$$G(z) = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} z+0.5 & 1 \\ 0.75 & z-0.5 \end{bmatrix}}{z^2-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ 4 puan

$$G(z) = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} z+0.5 \\ 0.75 \end{bmatrix}}{z^2-1} = \frac{0.75}{z^2-1}$$ 3 puan

(b)

$$Y(z) = G(z)\,U(z) \qquad u_k = (1 \; -1 \; 1 \; -1 \; \ldots)$$

$$Y(z) = \frac{0.75}{(z^2-1)} \qquad U(z) = \frac{z}{z-(-1)} = \frac{z}{z+1}$$ 3

$$Y(z) = 0.75 \frac{z}{(z+1)^2(z-1)}$$ 2

$$Y(z) = 0.75 \frac{z}{(z+1)^2(z-1)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{(z-1)}$$

$$0.75z = A(z-1)(z+1) + B(z-1) + C(z+1)^2$$

let $z=1$ $\quad 0.75 = 4C \quad C = \frac{\frac{3}{4}}{4} = \frac{3}{16}$

let $z=-1 \Rightarrow -0.75 = B(-2) \quad B = \frac{-0.75}{-2} = \frac{3}{8}$

let $z=0 \Rightarrow 0 = -A - B + C$

$$A = C - B = \frac{3}{16} - \frac{3}{8} = -\frac{3}{16}$$

$$Y(z) = -\frac{3}{16}\frac{1}{z+1} + \frac{3}{8}\frac{1}{(z+1)^2} + \frac{3}{16}\frac{1}{z-1}$$

$$= -\frac{3}{16}\frac{z^{-1}}{1+z^{-1}} + \frac{3}{8}\frac{z^{-2}}{(1+z^{-1})^2} + \frac{3}{16}\frac{z^{-1}}{1-z^{-1}}$$

$n a^n u(n) \xrightarrow{z} \frac{az}{(z-a)^2}$

$n(-1)^n u(n) \xrightarrow{z} \frac{-1z}{(z+1)^2}$

$-n(-1)^n u(n) \rightarrow \frac{z}{(z+1)^2}$

$$= -\frac{3}{16}(-1)^{k-1}\sigma_{k-1} - \frac{3}{8}(-1)^{k-1}\sigma_{k-1}(k-1) + \frac{3}{16}(1)^{k-1}\sigma_{k-1}$$

$$= \frac{3}{16}\left[(1)^{k-1} - (-1)^{k-1}\right]\sigma_{k-1} - \frac{3}{8}(k-1)(-1)^{k-1}\sigma_{k-1}$$

c) Not BIBO stable 5 poles over unit circle

(d) $x_{k+1} = \begin{bmatrix} 0.5 & 1 \\ 0.75 & -0.5 \end{bmatrix} x_k$

$$A_d = \begin{bmatrix} 0.5 & 1 \\ 0.75 & -0.5 \end{bmatrix}$$

$$A_d^2 = \begin{bmatrix} 0.5 & 1 \\ 0.75 & -0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0.75 & -0.5 \end{bmatrix}$$

$$A_d^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_d^3 = \begin{bmatrix} 0.5 & 1 \\ 0.75 & -0.5 \end{bmatrix}$$

$$A_d^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} a \\ b \end{bmatrix} \quad x_1 = \begin{bmatrix} 0.5a+b \\ 0.75a-0.5b \end{bmatrix} \quad x_2 = \begin{bmatrix} a \\ b \end{bmatrix} \quad x_3 = \begin{bmatrix} 0.5a+b \\ 0.75a-0.5b \end{bmatrix} \quad x_4 = \begin{bmatrix} a \\ b \end{bmatrix}$$

there is oscillation in the states

if k = even $\left\{ x_k = \begin{bmatrix} a \\ b \end{bmatrix} \right.$

if k = odd $\left\{ x_k = \begin{bmatrix} 0.5a+b \\ 0.75a-0.5b \end{bmatrix} \right.$

S

The system S is stable but not asymptotically stable