29-05-2016

MECE480 Final

Q1) A discrete time system is given by the difference equations as follows:

$$x_{k+1} = \begin{bmatrix} 0 & -0.1 \\ -0.4 & 0 \end{bmatrix} x_k + \begin{bmatrix} a \\ b \end{bmatrix} u_k, \qquad y_k = \begin{bmatrix} 4 & 1 \end{bmatrix} x_k$$

a) What are the eigenvalues of the system? (5 points)
b) What is the condition such that the system is completely controllable? (10 points)
c) Find the transfer function of this system? (15 points)
d) If a=0 and b=1, is it possible to replace the poles to any desired locations by state feedback? Use pole placement and try to replace them to locations z_1=0 and z_2=0. Find the parameters of state feedback vector if possible. (10 points)
e) If a=1 and b=2, is it possible to replace the poles to any desired locations by state feedback? Use pole placement and try to replace them to locations z_1=0 and z_2=0. Find the parameters of state feedback vector if possible. (10 points)

differential

Q2) A continuous time system is given by the ~~difference~~ equations as follows:

$$\dot{x} = \begin{bmatrix} -200 & 0 \\ 0 & -200 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

This system is discretized by a sampling period T=0.01 seconds, however while giving the discretized input to discretized system a delay occurs with respect to the actual samples. The delay is 0.005 seconds ($\Delta=$ 0.005 seconds).

Obtain the discrete state-space (ABCD) representation of the delayed input system. (25 points)

Q3) For a digital feedback control system the transfer function of the plant is given by the formula $G(z) = \frac{z+0.5}{(z-1)(z+1)}$. We want to control this plant by a controller C(z) such that the closed loop system will have poles at the location z_1=0, z_2=0 and z_3=0. Besides we want to obtain a 0 steady state error value for a reference of unit step input at the sampling time instances after 3 sampling instant has passed away. Find the controller structure C(z). (25 points)

MECE 480 Final solution

Q1 $x_{k+1} = \underbrace{\begin{bmatrix} 0 & -0.1 \\ -0.4 & 0 \end{bmatrix}}_{A_d} x_k + \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{b_d} u_k \qquad y_k = \underbrace{[4\ 4]}_{c_d^T} x_k$

5 point(1)

(a) eigenvalues $\Rightarrow \det(zI - A_d) = \det\left(\begin{bmatrix} z & 0.1 \\ 0.4 & z \end{bmatrix}\right) \Rightarrow z^2 - (0.1)(0.4) = 0$

$z_1 = 0.2 \rightarrow$ eigenvalues

$z_2 = -0.2$

(b) $\overset{\text{controllability matrix}}{Q} = \begin{bmatrix} b_d & A_d b_d \end{bmatrix} = \begin{bmatrix} a & -0.1b \\ b & -0.4a \end{bmatrix}$

for controllability $\operatorname{rank}(Q) = 2$

~~$a \neq 0 \quad -0.1a$~~

$a \neq 0 \quad -0.4 \neq 0$

$\det(Q) = a(-0.4a) - (b)(-0.1b) \neq 0$

$-0.4a^2 + 0.1b^2 \neq 0$

$0.1b^2 \neq 0.4a^2$

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$b^2 \neq 4a^2 \qquad b \neq 2a$

$b \neq -2a$

for controllability

$a \neq 0 \quad -0.4a \neq 0 \rightarrow a \neq 0$

$b \neq 2a$ and $b \neq -2a$ conditions should be satisfied

3

(c) $H(z) = c_d^T \left[zI - A_d\right]^{-1} b_d$

$= [4\ 4]\left(\begin{bmatrix} z & 0.1 \\ 0.4 & z \end{bmatrix}\right)^{-1} \begin{bmatrix} a \\ b \end{bmatrix}$ 7

$= [4\ 4] \dfrac{\begin{bmatrix} z & -0.1 \\ -0.4 & z \end{bmatrix}}{z^2 - 0.04} \begin{bmatrix} a \\ b \end{bmatrix} = \dfrac{[4z - 0.4 \quad z - 0.4]}{(z^2 - 0.04)} \begin{bmatrix} a \\ b \end{bmatrix}$ 8

$= \dfrac{(4a+b)z - 0.4(a+b)}{(z-0.2)(z+0.2)}$ 2

$$H(z) = (4a+b)\,\frac{z - \frac{0.4(a+b)}{4a+b}}{(z-0.2)(z+0.2)}$$

(d) If $a=0$ and $b=1$ $\quad Q = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix}$ $\quad$ rank$(Q) = 2$

completely controllable

3

hence ~~state~~-feedback replace all the poles to desired locations

$$A_{d.new} = A_d + b_d k_d^T \qquad k_d^T = (k_1 \ \ k_2)$$

$$= \begin{bmatrix} 0 & -0.1 \\ -0.4 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \ \ k_2]$$

$$A_{d\text{-}new} = \begin{bmatrix} 0 & -0.1 \\ -0.4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 \\ k_1-0.4 & k_2 \end{bmatrix}$$

$$zI - A_{d.new} = \begin{bmatrix} z & 0.1 \\ 0.4-k_1 & z-k_2 \end{bmatrix}$$

$$\det(zI - A_{d\,new}) = z(z-k_2) - (0.1)(0.4-k_1)$$

$$= z^2 - k_2 z - (0.1)(0.4-k_1)$$

4

new pole locations $\rightarrow p(z) = (z-z_1)(z-z_1) = z^2$

$0 \quad 0$

3

$$\det(zI - A_{d\text{-}new}) = z^2 = z^2 - k_2 z - (0.1)(0.4-k_1)$$

if $k_2 = 0$ $\ k_1 = 0.4$ equation is satisfied

$$k_d = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

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(e) $a=1 \quad b=2$

$$Q = \begin{bmatrix} 1 & -0.2 \\ 2 & -0.4 \end{bmatrix} \qquad \operatorname{rank}(Q) = 1 \quad \text{(not completely controllable)}$$

3

$$A_{d\text{-}new} = \begin{bmatrix} 0 & -0.1 \\ -0.4 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} (k_1 \quad k_2)$$

$$A_{d\text{-}new} = \begin{bmatrix} 0 & -0.1 \\ -0.4 & 0 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix} = \begin{bmatrix} k_1 & k_2-0.1 \\ 2k_1-0.4 & 2k_2 \end{bmatrix}$$

$$\det(zI - A_{dnew}) = \det\left(\begin{bmatrix} z-k_1 & 0.1-k_2 \\ 0.4-2k_1 & z-2k_2 \end{bmatrix}\right)$$

$$= (z-k_1)(z-2k_2) - (0.1-k_2)(0.4-2k_1)$$

$$= z^2 + (-k_1-2k_2)z + 2k_1k_2 - [0.04 + 2k_1k_2 - 0.4k_2 - 0.2k_1]$$

$$= z^2 - [k_1+2k_2]z + 2k_1k_2 - 0.04 - 2k_1k_2 + 0.4k_2 + 0.2k_1$$

$$= z^2 - [k_1+2k_2]z + [0.4k_2 + 0.2k_1 - 0.04] \overset{?}{=} z^2$$

4

$$p(z) = (z-z_1)(z-z_2) = z^2 \qquad p(z) \overset{?}{=} \det(zI - A_{dnew})$$

$z_1, z_2 \to 0$

3

$-[k_1+2k_2] = 0 \Rightarrow k_1+2k_2 = 0$

and

$0.4k_2 + 0.2k_1 = 0.04$

$0.2[k_1+2k_2] = 0.04$

$0.2 \times 0 = 0.04$ [not possible]

using state feedback we cannot locate the poles to $z_1=0$ and $z_2=0$

(Q2)

$$\dot{x} = \overbrace{\begin{bmatrix} -200 & 0 \\ 0 & -200 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^{b} u \qquad y = \underbrace{[1 \;\; 2]}_{c^T} x$$

$T = 0.01$ sec

$\Delta = 0.005$ sec

$$[sI - A] = \begin{bmatrix} s+200 & 0 \\ 0 & s+200 \end{bmatrix} \Rightarrow e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+200} & 0 \\ 0 & \frac{1}{s+200} \end{bmatrix}\right\}$$

$$e^{At} = \begin{bmatrix} e^{-200t} & 0 \\ 0 & e^{-200t} \end{bmatrix}$$

$$A_d = e^{AT}$$

$$A_d = e^{AT} = e^{A \times 0.01}$$

$$= e^{A \times 0.01} = \begin{bmatrix} e^{-2} & 0 \\ 0 & e^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & \frac{1}{e^2} \end{bmatrix}$$

5 point

$$\tilde{b}_d = \int_0^{\Delta} e^{A\tau} b \, d\tau = \int_0^{0.005} \begin{bmatrix} e^{-200\tau} & 0 \\ 0 & e^{-200\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} d\tau$$

$$= \int_0^{0.005} \begin{bmatrix} e^{-200\tau} \\ 2e^{-200\tau} \end{bmatrix} d\tau = \begin{bmatrix} -\frac{1}{200} e^{-200\tau} \Big|_0^{0.005} \\ 2\left(-\frac{1}{200}\right) e^{-200\tau} \Big|_0^{0.005} \end{bmatrix}$$

$$\tilde{b}_d = \begin{bmatrix} -\frac{1}{200}\left(e^{-1} - 1\right) \\ -\frac{1}{100}\left[e^{-1} - 1\right] \end{bmatrix} = \begin{bmatrix} \frac{1}{200}\left[1 - \frac{1}{e}\right] \\ \frac{1}{100}\left[1 - \frac{1}{e}\right] \end{bmatrix}$$

5 point

$$b_d = \int_A^T e^{A\tau} b \, d\tau = \int_{0.005}^{0.01} \begin{bmatrix} e^{-200\tau} & 0 \\ 0 & e^{-200\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} d\tau$$

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$$= \int_{0.005}^{0.01} \begin{bmatrix} e^{-200\tau} \\ 2e^{-200\tau} \end{bmatrix} d\tau$$

5 point

$$= \begin{bmatrix} -\frac{1}{200} e^{-200\tau} \Big|_{0.005}^{1} \\ 2\left(-\frac{1}{200}\right) e^{-200\tau} \Big|_{0.005}^{1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{200}\left[\frac{1}{e^2} - \frac{1}{e}\right] \\ -\frac{1}{100}\left[\frac{1}{e^2} - \frac{1}{e}\right] \end{bmatrix}$$

$$b_d = \begin{bmatrix} \frac{1}{200}\left[\frac{1}{e} - \frac{1}{e^2}\right] \\ \frac{1}{100}\left[\frac{1}{e} - \frac{1}{e^2}\right] \end{bmatrix}$$

Gidiş 8

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A_d & \check{b}_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} + \begin{bmatrix} b_d \\ 1 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} c_d^T & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}$$

2 point

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{e^2} & 0 & \frac{1}{200}\left(1 - \frac{1}{e}\right) \\ 0 & \frac{1}{e^2} & \frac{1}{100}\left[1 - \frac{1}{e}\right] \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} + \begin{bmatrix} \frac{1}{200}\left[\frac{1}{e} - \frac{1}{e^2}\right] \\ \frac{1}{100}\left[\frac{1}{e} - \frac{1}{e^2}\right] \\ 1 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}$$

Q3

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$$G(z) = \frac{(z+0.5)}{[z-1][z+1]} \qquad Q(z) = z^3$$

[$p = 3$ after 3 steps the error will be 0]

$$C(z) = \frac{p_0 + p_1 z}{l_0 + l_1 z}$$

3 point

$$1 + G(z)C(z) = 1 + \frac{(z+0.5)}{[z-1][z+1]} \frac{(p_0 + p_1 z)}{(l_0 + l_1 z)}$$

$$1 + G(z)C(z) = \frac{[z-1][z+1](l_0 + l_1 z) + [z+0.5](p_0 + p_1 z)}{[z-1][z+1](l_0 + l_1 z)} =$$

5 point

$$Q(z) = \text{roots of } \left[1 + G(z)C(z)\right] = [z^2 - 1][l_0 + l_1 z] + (z+0.5)(p_0 + p_1 z)$$

$$z^3 = [z^2 - 1](l_0 + l_1 z) + (z + 0.5)(p_0 + p_1 z)$$

5 point

5

$$z^3 = l_1 z^3 + l_0 z^2 - l_1 z - l_0 + p_1 z^2 + [p_0 + 0.5 p_1] z + 0.5 p_0$$

$$z^3 = l_1 z^3 + [l_0 + p_1] z^2 + [p_0 + 0.5 p_1 - l_1] z + [0.5 p_0 - l_0]$$

$l_1 = 1 \qquad l_0 + p_1 = 0 \qquad p_0 + 0.5 p_1 - l_1 = 0 \qquad 0.5 p_0 - l_0 = 0$

$p_1 = -l_0 \qquad p_0 + 0.5 p_1 = 1 \qquad p_0 = 2 l_0$

Grade is 12

$$2 l_0 + 0.5(-l_0) = 1$$

$$\frac{3}{2} l_0 = 1 \qquad \boxed{l_0 = \frac{2}{3}}$$

$p_1 = -l_0 = -\frac{2}{3}$ $\qquad p_0 = 2l_0 = 2\times\frac{2}{3} = \frac{4}{3}$

6 point

$$C(z) = \frac{p_0 + p_1 z}{l_0 + l_1 z} = \frac{\frac{4}{3} + \left(-\frac{2}{3}\right)z}{\frac{2}{3} + 1z} = \frac{-\frac{2}{3}\left(-\frac{2}{1} + z\right)}{\frac{2}{3} + z}$$

$$C(z) = -\frac{2}{3}\,\frac{[z-2]}{z+\frac{2}{3}}$$

Let's find error

$$E(z) = \frac{U(z)}{1 + G(z)C(z)} = \frac{1}{1 + \frac{(z+0.5)}{(z-1)(z+1)}\left[-\frac{2}{3}\right]\frac{[z-2]}{\left[z+\frac{2}{3}\right]}}\,U(z)$$

$$E(z) = \frac{3[z-1][z+1]\left(z+\frac{2}{3}\right)}{3(z-1)(z+1)\left[z+\frac{2}{3}\right] - (2)(z+0.5)(z-2)}\,U(z)$$

$$= \frac{3[z-1][z+1]\left[z+\frac{2}{3}\right]}{3[z^2-1]\left[z+\frac{2}{3}\right] - 2\left[z^2-1.5z-1\right]}\,U(z)$$

$$E(z) = \frac{3(z-1)(z+1)\left(z+\frac{2}{3}\right)}{3\left[z^3+\frac{2}{3}z^2 - z - \frac{2}{3}\right] - 2\left(z^2 - 1.5z - 1\right]}\,U(z)$$

$$E(z) = \frac{3z^3 + 2z - 3z - 2}{3z^2 + \cancel{8}2z - 3z - 2 - 2z^2 + 3z + 2} U(z)$$

$$E(z) = \frac{3z^2 + 2z - 3z - 2}{z^3}$$

$q_r = \sigma_r$
↓
unit step

$$U(z) = \frac{z}{z-1} \Rightarrow E(z) = \frac{3z^2 + 3z - 3z - 2}{z^3} = \frac{3(z-1)(z+1)(z+\frac{2}{3})}{z^3} \frac{z}{z-1}$$

$$E(z) = \frac{3(z+1)(z+\frac{2}{3})}{z^2} = \frac{3\left[z^2 + \frac{5}{3}z + \frac{2}{3}\right]}{z^2} = \frac{z^2 + 5z + 2}{z^2}$$

$$E(z) = 1 + 5z^{-1} + 2z^{-2}$$

↓

$$e_k = [1 \quad 5 \quad 2 \quad 0 \quad 0 \quad \ldots \quad]$$

↓
samples of error