MECE480 Final
31-05-2016
Tuesday

Q1) A digital system is given by the formula

$$x_{k+1} = \begin{bmatrix} -0.5 & 10 \\ 0 & -0.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \text{ and } y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k$$

a) Find the transfer function of this digital system
b) Show that this system is not completely controllable.
c) Show that this system is observable.
d) Find the eigenvalues of this digital system. And use the Hautus test for finding the eigenvalue, which is not controllable.
e) Even if it is not completely controllable, it is possible for this system to move one of the eigenvalues to z=0 by state feedback. Show the possibility of this process.

Q2) An open loop digital system is given by the open loop transfer function $G_o(z) = \frac{K}{z^3}$. For which K values this system becomes unstable (poles of the system passes the unit circle outwardly).

Q3) For a digital feedback control system the transfer function of the plant is given by the formula: $G(z) = \frac{z-3}{z(z+0.5)}$

We want to control this plant by a controller C(z) such that the closed loop system will have poles at the locations $z_1=0$, $z_2=0$ and $z_3=-0.1$. Find the controller structure C(z).

Q4) For a digital feedback control system the transfer function of the plant is given by the formula: $G(z) = \frac{z-2}{(z+1)}$

We want to control this plant by a controller C(z) such that the closed loop system will have poles at the locations $z_1=0$, $z_2=0$ and $z_3=0$. Besides we want to obtain a zero steady state error value for a reference of unit step input at the sampling times instants after 3 sampling instant has passed away. Find the controller structure C(z).

MECE 480 Final
2015–2016 spring

Q1

$$A = \begin{bmatrix} -0.5 & 10 \\ 0 & -0.5 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad c^T = [1\ 1]$$

(a) $T(z) = c^T [zI - A]^{-1} b$

$$= [1 \quad 1] \begin{bmatrix} z+0.5 & -10 \\ 0 & z+0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(z) = \frac{1 \cdot [1 \quad 1] \begin{bmatrix} z+0.5 & 10 \\ 0 & z+0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{[z+0.5]^2}$$

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$$T(z) = \frac{[1 \quad 1] \begin{bmatrix} z+0.5 \\ 0 \end{bmatrix}}{(z+0.5)^2} = \frac{z+0.5}{(z+0.5)^2} = \frac{1}{z+0.5}$$

(b) Controllability $Q = (b \quad Ab] = \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ 5 $\text{rank}(Q) = 1 \neq 2$ not full-rank

not completely controllable

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(c) Observability $O = \begin{bmatrix} c^T \\ c^T A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -0.5 & 9.5 \end{bmatrix}_{2\times 2}$ $\text{Rank}(O) = 2 = 2$

full-rank

(d) Hautus test $\det(\lambda I - A) = (\lambda + 0.5)^2 = 0 \to \lambda_1 = 0.5,\ \lambda_2 = 0.5$

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$$[\lambda I - A \ \ b] = \begin{bmatrix} \lambda+0.5 & 10 & 1 \\ 0 & \lambda+0.5 & 0 \end{bmatrix}$$

put. $\lambda = 0.5$ $= \begin{bmatrix} 0 & 10 & 1 \\ 0 & 0 & 0 \end{bmatrix} = K$ $\quad \text{rank}(K) = 1 \to$ not full rank

$\lambda_1 = 0.5$

$\lambda_1 = 0.5 \rightarrow$ not-controllable eigenvalue

(e) $A + bk^T = \tilde{A} = \begin{bmatrix} -0.5 & 10 \\ 0 & -0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \ k_2]$

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$$\tilde{A} = \begin{bmatrix} k_1 - 0.5 & k_2 + 10 \\ 0 & -0.5 \end{bmatrix}$$

$$[zI - \tilde{A}] = \begin{bmatrix} z + 0.5 - k_1 & -[10 + k_2] \\ 0 & z + 0.5 \end{bmatrix}$$

$$\det[zI - \tilde{A}] = [z + 0.5][z + 0.5 - k_1] - (0)(-(10 + k_2))$$

$$= [z + 0.5][z + 0.5 - k_1] = (z + 0.5)(z + 0)$$

$0.5 - k_1 = 0 \quad k_1 = 0.5$ should be taken

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ \text{arbitrary} \end{bmatrix}$$

(Q2) $G_0(z) = \dfrac{K}{z^3}$ $\qquad 1 + G_0(z) = 1 + \dfrac{K}{z^3} = 0$ $\qquad$ put $z = a + bj$

$|z| = \sqrt{a^2 + b^2} = 1$ [for crossing unit circle]

$$1 + \frac{K}{[a + bj]^3} = 0$$

5 $[a + bj]^3 + K = 0$

$[a^2 - b^2 + 2abj][a + bj] + K = 0$

$a^3 - ab^2 + 2a^2bj + a^2bj - b^3j + 2ab^2j^2 + K = 0$

$[a^3 - ab^2 - 2ab^2] + j[2a^2b - b^3 + a^2b] + K = 0$

5 $[a^3 - 3ab^2] + j[3a^2b - b^3] + K = 0$

$$\left.\begin{aligned} a^3-3ab^2+K&=0\\ 3a^2b-b^3&=0\\ a^2+b^2&=1 \end{aligned}\right\}$$ equations

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$b\left[3a^2-b^2\right]=0 \rightarrow$ either $b=0$
or $b^2=3a^2$

* $b=0 \Rightarrow a^2=1 \quad a^3=-K$

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$a=1 \Rightarrow a^3=1=-K \Rightarrow K=-1$ (negative gain $\rightarrow$ not in root locus)

$a=-1 \Rightarrow a^3=-1=-K \Rightarrow K=1$ (positive gain $\rightarrow$ in root locus)

solution 1 $\quad b=0, a=-1, K=1$

* if $3a^2=b^2 \Rightarrow a^2+b^2=1$

$a^2+3a^2=1$

$4a^2=1$

$a=\mp\frac{1}{2}$

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$a^3-3ab^2+K=0$

$a^3-3a(3a^2)+K=0$

$a^3-9a^3+K=0$

$K=8a^3 \rightarrow$ positive for $a=\frac{1}{2}$

$\boxed{a=\frac{1}{2} \Rightarrow K=1 \quad b=\mp\frac{\sqrt{3}}{2}} \rightarrow$ positive K

$a=-\frac{1}{2} \Rightarrow K=-1 \quad b=\mp\frac{\sqrt{3}}{2}$

$\boxed{\text{when } K=1 \quad z_1=\frac{1}{2}+\frac{\sqrt{3}}{2}j,\ z_2=\frac{1}{2}-\frac{\sqrt{3}}{2}j,\ z_3=-1+0j}$

Q2

$$G(z) = \frac{K}{(z+0.3)(z-0.5)} \qquad 1+G(z) = 1+\frac{K}{(z+0.3)(z-0.5)} = 0$$

$$(z+0.3)(z-0.5)+K=0$$
$$z^2+(-0.2)z-0.15+K=0$$
$$z^2-0.2z-0.15+K=0$$

(a) If the poles of closed loop system are over the unit circle or outside the unit circle the system becomes unstable.

$$\cancel{|z|=1} \qquad z = 1e^{j\theta} = a+bj \qquad a^2+b^2=1$$

$$e^{j2\theta} - 0.2e^{j\theta} - 0.15+K=0$$
$$[a+bj]^2 - 0.2[a+bj]+K-0.15=0$$
$$a^2-b^2+2abj-0.2a-0.2bj+K-0.15=0$$
$$[a^2-b^2-0.2a+K-0.15]=0 \qquad \underbrace{[2ab-0.2b]}_{0}j=0$$

$$2ab-0.2b=0 \qquad b.[2a-0.2]=0 \qquad a=0.1$$

$$a^2+b^2=1 \rightarrow b^2=1-0.01 \qquad b=\sqrt{0.99}$$

$$a^2-b^2-0.2a+K-0.15=0$$
$$0.01-0.99-0.2\times0.1+K-0.15=0$$
$$K=0.15+0.99+0.02-0.01$$
$$K=0.15+1=1.15$$

if $K>1.15$ system is unstable

Q3 $G(z) = \frac{z-3}{z(z+0.5}$ relocate the poles to $z_1 = 0,\ z_2 = 0,\ z_3 = -0.1$

$Q(z) = z^2(z+0.1) = z^3 + 0.1z^2$ // 5

$$1 + G(z)C(z) = 1 + \frac{z-3}{z(z+0.5)} \frac{(p_0 + p_1 z)}{[l_0 + l_1 z]} = z^2(z+0.1) = z^3 + 0.1z^2$$

5 $l_1 z^3 + [0.5l_1 + l_0 + p_1]z^2 + [0.5l_0 + p_0 - 3p_1]z - 3p_0 = z^3 + 0.1z^2$

$l_1 = 1 \quad p_0 = 0 \quad 0.5l_1 + l_0 + p_1 = 0.1 \quad 0.5l_0 + p_0 - 3p_1 = 0$

$0.5 + l_0 + p_1 = 0.1 \qquad 0.5l_0 - 3p_1 = 0$

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$l_0 + p_1 = -0.4 \qquad l_0 = 6p_1$

$+ \qquad 6p_1 + p_1 = -0.4 \qquad p_1 = \frac{-0.4}{7}$ //

$l_0 = \frac{-2.4}{7}$ //

$$C(z) = \frac{p_0 + p_1 z}{l_0 + l_1 z} = \frac{-\frac{0.4}{7}z}{\frac{-2.4}{7} + z} \ // = \frac{-0.4z}{7z - 2.4} \ //$$

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Q4

$$G(z) = \frac{(z-2)}{z+1}$$

Dead-beat control with closed loop poles at $z_1=0, z_2=0, z_3=0$

$Q(z) = z^3$ [$p=3$ after 3 steps the error will be 0]

$$C(z) = \frac{\rho_0 + \rho_1 z}{(\ell_0' + \ell_1' z)(z-1)}$$ 5

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$$1 + G(z)C(z) = 1 + \frac{(z-2)}{[z+1]} \frac{[\rho_0 + \rho_1 z]}{(\ell_0' + \ell_1 z)(z-1)}$$

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$$[z+1][z-1][\ell_0' + \ell_1 z] + [z-2][\rho_0 + \rho_1 z] = z^3$$

$$[z^2 - 1]\{[\ell_0' + \ell_1' z] + [\rho_1 z^2 + [\rho_0 - 2\rho_1] z - 2\rho_0 = z^3$$

$$\ell_1' z^3 + (\ell_0' z^2] - \ell_1' z - \ell_0' + (\rho_1 z^2 + (\rho_0 - 2\rho_1] z - 2\rho_0] = z^3$$

$$\ell_1' z^3 + [\rho_1 + \ell_0'] z^2 + [\rho_0 - 2\rho_1 - \ell_1'] z - [\ell_0' + 2\rho_0] = z^3$$

$\ell_1' = 1$ (3) $\quad \boxed{\rho_1 + \ell_0' = 0} \quad \rho_0 - 2\rho_1 - \ell_1' = 0 \quad [\ell_0' + 2\rho_0] = 0$

$\boxed{\rho_0 - 2\rho_1 = 1}$

$\rho_1 = -\ell_0' \qquad \ell_0' = -2\rho_0 \qquad \rho_0 = -\frac{1}{3}$ // 3

$\rho_1 = 2\rho_0 \qquad \rho_0 - 2\rho_1 = 1 \qquad \rho_1 = -\frac{2}{3}$ // 3

$\rho_0 - 2 \times 2\rho_0 = 1$

$-3\rho_0 = 1 \qquad \ell_0' = \frac{2}{3}$ // 3

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$$C(z) = \frac{-\frac{1}{3} - \frac{2}{3} z}{\left[\frac{2}{3} + z\right](z-1)} = \frac{-1-2z}{2+3z} = \frac{-(1+2z)}{2+3z}$$ //