2015-2016 spring MECE 480 homework 7 solutions

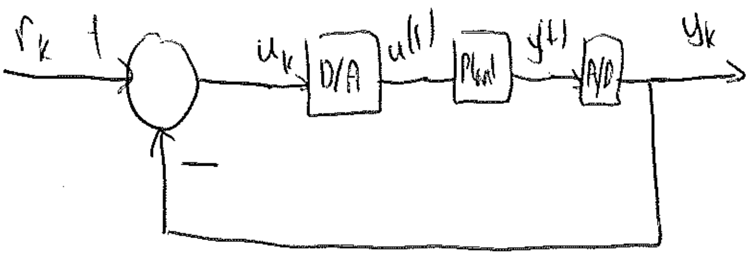
Q1

reference sequence Page 1

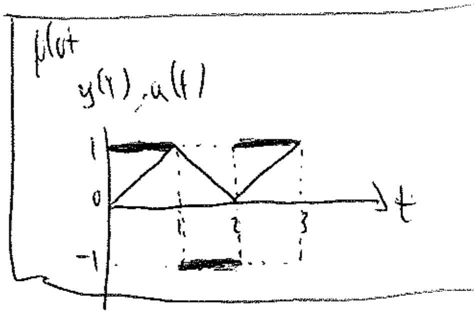
$$y(t) = \int_0^t u(\tau)\,d\tau \qquad r_k = [1, 0, 1,]$$

Control algorithm $u_k = (r_k - y_k)$ Sampling time $T = 1$

D/A - converter: zero order hold



$y(0) = 0$



$r_0 = 1 \quad y_0 = 0$

$0 \leq t < 1$

$u_0 = (r_0 - y_0) = (1 - 0) = 1$ hence $u(t) = 1\sigma(t) \quad 0 \leq t < 1$

$$y(t) = \int_0^t u(\tau)\,d\tau = \int_0^t 1\,d\tau = \tau \Big|_0^t = t$$

$y(0) = 0$
$y(1) = 1$

$1 \leq t < 2$

$r_1 = 1 \quad y_1 = 1$

$u_1 = (0 - 1) = -1$ hence $u(t) = -1\sigma(t-1) \quad 1 \leq t < 2$

$$y(t) = \int_0^t u(\tau)\,d\tau = \int_0^1 1\,d\tau + \int_1^t (-1)\,d\tau = \tau\Big|_0^1 - \tau\Big|_1^t = 1 - (t-1)$$
$$= 2 - t$$

$y(t) = 2 - t \quad 1 \leq t < 2 \quad y(1) = 1, \; y(2) = 0$

$2 \leq t < 3$

$r_2 = 1 \quad y_2 = 0 \quad u_2 = (1 - 0) = 1$ hence $u(t) = 1\sigma(t-2) \quad 2 \leq t < 3$

$$y(t) = \int_0^t u(\tau)\,d\tau = \int_0^1 \underbrace{u(\tau)}_{1}\,d\tau + \int_1^2 \underbrace{u(\tau)}_{(-1)}\,d\tau + \int_2^t \underbrace{u(\tau)}_{1}\,d\tau$$

$$= \tau\Big|_0^1 + (-\tau)\Big|_1^2 + \tau\Big|_2^t = 1 - (2-1) + [t-2] = t - 2$$

$y(3) = 1$

Q2

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(a) $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $A_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $b_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad zI - A_d = \begin{bmatrix} z & -1 \\ -1 & z \end{bmatrix}$

$c_d^T = [0 \ 1]$

$X(z) = (zI - A_d)^{-1} b_d U(z)$ $\quad [zI - A_d]^{-1} = \dfrac{\begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix}}{z^2 - 1}$

$X(z) = \begin{bmatrix} \frac{1}{z^2-1} \\ \frac{z}{z^2-1} \end{bmatrix} U(z)$ $\quad u_k = (1, 1, 1, \dots) \xrightarrow{Z} U(z) = \dfrac{z}{z-1}$

$$X(z) = \begin{bmatrix} \frac{z}{(z-1)^2(z+1)} \\ \frac{z^2}{(z-1)^2(z+1)} \end{bmatrix} = \begin{bmatrix} z^{-1}\left[\frac{3}{4}\frac{1}{z-1} + \frac{1}{2}\frac{1}{(z-1)^2} + \frac{1}{4}\frac{1}{z+1}\right] \\ \left[\frac{3}{4}\frac{1}{z-1} + \frac{1}{2}\frac{1}{(z-1)^2} + \frac{1}{4}\frac{1}{z+1}\right] \end{bmatrix}$$

$$\frac{z^2}{(z-1)^2(z+1)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1}$$

$z^2 = A(z-1)(z+1) + B(z+1) + C(z-1)^2$

$z^2 = [A + C]z^2 + [B - 2C]z + [B + C - A]$

$A + C = 1 \quad B = 2C \quad B + C = A$

$A = \frac{3}{4} \quad B = \frac{1}{2} \quad C = \frac{1}{4}$

$$x_k = \begin{bmatrix} \frac{3}{4}\sigma_{k-2} + \frac{1}{2}(k-2)\sigma_{k-2} + \frac{1}{4}(-1)^{k-2}\sigma_{k-2} \\ \frac{3}{4}\sigma_{k-1} + \frac{1}{2}(k-1)\sigma_{k-1} + \frac{1}{4}(-1)^{k-1}\sigma_{k-1} \end{bmatrix}$$

$y_k = [0 \ 1] x_k$

$y_k = \frac{3}{4}\sigma_{k-1} + \frac{1}{2}(k-1)\sigma_{k-1} + \frac{1}{4}(-1)^{k-1}\sigma_{k-1}$

(b) $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad u_k = (0, 0, 0, \dots)$

$X(z) = z(zI - A_d)^{-1} x_0 = \begin{bmatrix} \frac{z}{z^2-1} \\ \frac{z^2}{z^2-1} \end{bmatrix}$

$\dfrac{z^2}{z^2-1} = 1 + \dfrac{1}{z^2-1} = 1 + \dfrac{A}{z-1} + \dfrac{B}{z+1}$

$= 1 + \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$

(Q2) (b) continue

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$$X(z) = \begin{bmatrix} z^{-1}\left[1 + \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}\right] \\ \left[1 + \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}\right] \end{bmatrix}$$

$$x_k = \begin{bmatrix} \delta_{k-1} + \frac{1}{2}\sigma_{k-2} - \frac{1}{2}(-1)^{k-2}\sigma_{k-2} \\ \delta_k + \frac{1}{2}\sigma_{k-1} - \frac{1}{2}(-1)^{k-1}\sigma_{k-1} \end{bmatrix}$$

$$y_k = [0 \; 1]\, x_k$$

$$\boxed{y_k = \delta_k + \frac{1}{2}\sigma_{k-1} - \frac{1}{2}(-1)^{k-1}\sigma_{k-1}}$$

Q3 (a) $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u_k = \sigma_k$ $A_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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$$x_k = \sum_{j=0}^{k-1} A_d^{k-1-j} b_d u_j$$

$$x_k = A_d^{k-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 + A_d^{k-2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 + \dots + A_d^{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + A_d^{1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + A_d^{0}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X_k = \left[A_d^{k-1} + A_d^{k-2} + \dots + A_d^{2} + A_d^{1} + A_d^{0}\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x_k = \left[A_d^{k-1} + A_d^{k-2} + \dots + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} A_d^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A_d^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ A_d^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A_d^{even} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A_d^{odd} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}}$$

$$x_k = \begin{cases} \begin{bmatrix} \frac{k}{2} \\ \frac{k}{2} \end{bmatrix} & \text{if } k = \text{even} \\ \begin{bmatrix} \frac{k-1}{2} \\ \frac{k+1}{2} \end{bmatrix} & \text{if } k = \text{odd} \end{cases}$$

$$y_k = [0 \; 1]\, x_k = \begin{cases} \frac{k}{2}, & \text{if } k = \text{even} \\ \frac{k+1}{2}, & \text{if } k = \text{odd} \end{cases}$$

(b) $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $u_k = (0, 0, 0, \dots)$

$y_k = c_d^T A_d^k x_0$ $x_k = A_d^k x_0$ $A_d^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A_d^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A_d^k = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & k = \text{even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & k = \text{odd} \end{cases} \qquad x_k = A_d^k x_0 = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } k = \text{even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } k = \text{odd} \end{cases}$$

$$x_k = \begin{bmatrix} \sigma_k \\ \sigma_k \end{bmatrix} \qquad y_k = [1 \; 1]\, x_k = 2\sigma_k$$

Q4

$$\frac{1}{(z-1)^2(z+1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

$$* \quad 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z-1)^2(z+1) + D(z-1)^2$$

put $z=1$ $\quad 1 = 4B \quad B = \frac{1}{4}$

put $z=-1$ $\quad 1 = 4D \quad D = \frac{1}{4}$

Take derivative of $*$

$$0 = A(z+1)^2 + 2(z+1)(z-1)A + B2(z+1) + C2(z-1)(z+1) + C(z-1)^2 + 2D(z-1)$$

put $z=1$ $\quad 0 = 4A + 4B \quad A = -B \quad A = -\frac{1}{4}$ //

put $z=-1$ $\quad 0 = 4C - 4D \quad C = D \quad D = \frac{1}{4}$

$$\frac{1}{(z-1)^2(z+1)^2} = -\frac{1}{4}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{4}\frac{1}{z+1} + \frac{1}{4}\frac{1}{(z+1)^2}$$

we know that

$a^k \sigma_k \xrightarrow{z} \frac{1}{1-az^{-1}} = \frac{z}{z-a}$ // $\qquad$!!!!!! important

$k a^k \sigma_k \xrightarrow{z} \frac{az^{-1}}{[1-az^{-1}]^2} = \frac{\frac{a}{z}}{[1-\frac{a}{z}]^2} = \frac{az}{[z-a]^2}$ //

$k(-1)^k \sigma_k \xrightarrow{z} \frac{-1z^{-1}}{[1+z^{-1}]^2} \qquad (k-1)(-1)^{k-1}\sigma_{k-1} \xrightarrow{z} \frac{-1z^{-1}z^{-1}}{\boxed{[1+z^{-1}]^2}}$

Q4 continue

$$F(z) = -\frac{1}{4}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{4}\frac{1}{z+1} + \frac{1}{4}\frac{1}{(z+1)^2}$$

$$F(z) = -\frac{1}{4}\frac{z^{-1}}{1-1z^{-1}} + \frac{1}{4}\frac{z^{-1}z^{-1}}{[1-1\cdot z^{-1}]^2} + \frac{1}{4}\frac{z^{-1}}{1+1z^{-1}} + \frac{1}{4}\frac{z^{-1}z^{-1}}{(1+z^{-1})^2}$$

$$\Big\downarrow \mathcal{Z}^{-1}$$

$$f_k = -\frac{1}{4}\sigma_{k-1} + \frac{1}{4}(k-1)\sigma_{k-1} + \frac{1}{4}(-1)^{k-1}\sigma_{k-1} + \frac{1}{4}(-1)((k-1)(-1)^{k-1}\sigma_{k-1}$$

$$f_k = -\frac{1}{4}\sigma_{k-1} + \frac{1}{4}(k-1)\sigma_{k-1} + \frac{1}{4}(-1)^{k-1}\sigma_{k-1} - \frac{1}{4}(k-1)(-1)^{k-1}\sigma_{k-1}$$

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Q5 $G(s) = \dfrac{1}{s(s+1)}$

$$G(z) = \frac{z-1}{z}\, Z\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\Big|_{t=kT}\right)$$

$$\frac{G(s)}{s} = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$1 = As(s+1) + B(s+1) + Cs^2 \qquad 1 = (A+C)s^2 + (A+B)s + B$$

$$B = 1 \qquad A = -1 \qquad C = 1$$

$$F(s) = \frac{G(s)}{s} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} f(t) = \left[-1 + t + e^{-t}\right]\sigma(t)$$

$$f(kT) = \left[-1 + kT + e^{-kT}\right]\sigma_k$$

$$f(kT) \xrightarrow{Z} \left[-\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}\right]$$

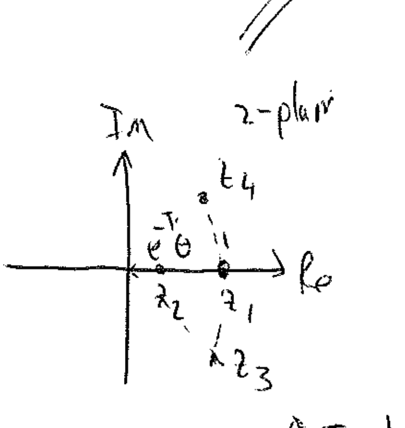
$$G(z) = \frac{z-1}{z}\left[-\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}\right] = \left[-1 + \frac{T}{z-1} + \frac{z-1}{ze^{-T}}\right]$$

Q6 $s_1 = 0 \quad s_2 = -1 \quad s_3 = -j \quad s_4 = j$

$z_1 = e^{s_1T} \quad z_2 = e^{s_2T} \quad z_3 = e^{s_3T} \quad z_4 = e^{s_4T}$

$z_1 = 1 \quad z_2 = e^{-T} \quad z_3 = e^{-jT} \quad z_4 = e^{jT}$

$z_3 = \cos(T) - j\sin T \qquad z_4 = \cos(T) + j\sin(T)$



$\theta = T\,\text{rad}$

Q7 $x_{k+1} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_d} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k$

(a) $zI - A_d = \begin{bmatrix} z & -1 \\ -1 & z \end{bmatrix} \Rightarrow z^2 - 1 = 0 = \det[zI - A_d]$

$z_1 = 1$, $z_2 = -1$ (poles of system over unit circle

(NOT AYMPTOTICALLY STABLE)

(b) Since poles are on the unit circle (each) the zero input solution is stable

(c) NOT BIBO stable [there are poles on the unit circle]

Q8 $$\frac{U(z)}{E(z)} = K_p \frac{(\gamma T_1 + 1)z + (1 - \gamma T_1)}{\gamma T_1 (z-1)}$$

$$U(z)\, \gamma T_1 [z - 1] = E(z)\, K_p \left[(\gamma T_1 + 1)z + (1 - \gamma T_1)\right]$$

devide by z

$$U(z)\, \gamma T_1 \left[1 - \frac{1}{z}\right] = E(z)\, K_p \left[(\gamma T_1 + 1) + \frac{(1 - \gamma T_1)}{z}\right]$$

$\Big\downarrow \mathcal{Z}^{-1}$

$$\gamma T_1 u_k - \gamma T_1 u_{k-1} = K_p(\gamma T_1 + 1) e_k + K_p(1 - \gamma T_1) e_{k-1}$$

$$\boxed{u_k = u_{k-1} + \frac{K_p(\gamma T_1 + 1)}{\gamma T_1} e_k + \frac{K_p(1 - \gamma T_1)}{\gamma T_1} e_{k-1}}$$