MECE480
Midterm
02-03-2016

Q1) The Z transform of a discrete sequence is given by the formula $F(z)=\dfrac{16}{(z-2)^2(z+2)^2}$ using partial fraction expansion, find the discrete time sequence f_k. **(15 points)**

Q2) A digital Control System is given with the following properties: Plant description: $y(t)=2\int_0^t u(\tau)d\tau$, Reference Sequence: $r_k=(1,0,-1)$, Control Algorithm: $u_k=(r_k-y_k)$, Sampling Time: T=1, D/A Converter: Zero-order hold

Assuming that y(0)=0 find u(t) and y(t) when $0\le t\le 3$ and draw them. **(25 points)**

Q3) A discrete time system state equation is given by the formula

$$x_{k+1}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}x_k+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u_k \text{ and } y_k=\begin{bmatrix} 0 & 1 \end{bmatrix}x_k$$

Assuming that $x_0=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $u_k=(-1)^k$ find x_k and y_k using Z transformation. **(30 points)**

Q4) A continuous time transfer function is given by the formula $G(s)=\dfrac{1}{s}$

Find the discrete time transfer function G(z). **(10 points)**

Q5) A discrete system is given by the formula:

$$x_{k+1}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}x_k+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u_k \text{ and } y=\begin{bmatrix} 0 & 1 \end{bmatrix}x_k$$

a) Is the zero input solution of this system asymptotically stable? Explain.
b) Is the zero input solution of this system stable? Explain.
c) Is this system BIBO stable? Explain

(10 points)

Q6) The Tustin transformation applied to a continuous time controller is given by the formula $\dfrac{U(z)}{E(z)}=C(z)=K_p\dfrac{(\gamma T_I+1)z+(1-\gamma T_I)}{\gamma T_I(z-1)}$. Find the discrete time control strategy u_k.

(10 points)

MECE 480 Midterm 2015-2016 spring

Q1

$$F(z) = \frac{16}{(z+2)^2(s-2)^2} = \frac{A}{(z+2)} + \frac{B}{(z+2)^2} + \frac{C}{(z-2)} + \frac{D}{(z-2)^2}$$

$$16 = A(z+2)(z-2)^2 + B(z-2)^2 + C(z+2)^2(z-2) + D(z+2)^2$$

put $z=2$ $\quad 16 = D16 \quad D=1$

put $z=-2$ $\quad 16 = B16 \quad B=1$

$$0 = A\left[(s-2)^2 + 2(s-2)(s+2)\right] + B2(s-2) + C\left[2(s+2)(s-2) + (s+2)^2\right] + D2(s+2)$$

put $z=2$

$0 = 16C + 8D \qquad C = -\frac{1}{2}$

put $z=-2$

$0 = 16A - 8B \qquad A = \frac{1}{2}$

10

$$F(z) = \frac{16}{(z+2)^2(z-2)^2} = \frac{1}{2}\frac{1}{z+2} + \frac{1}{(z+2)^2} + \left(-\frac{1}{2}\right)\frac{1}{z-2} + \frac{1}{(z-2)^2}$$

$$F(z) = \frac{1}{2}\frac{z^{-1}}{1+2z^{-1}} + \frac{z^{-1}z^{-1}}{(1+2z^{-1})^2} - \frac{1}{2}\frac{z^{-1}}{1-2z^{-1}} + \frac{z^{-1}z^{-1}}{(1-2z^{-1})^2}$$

5

$$\Big\downarrow z^{-1}$$

$$f(k) = \frac{1}{2}(-2)^{k-1}\sigma_{k-1} + \frac{1}{2}(-2)^{k-1}(k-1)\sigma_{k-1} - \frac{1}{2}2^{k-1}\sigma_{k-1} + \frac{1}{2}(k-1)2^{k-1}\sigma_{k-1}$$

(Q2) $r_k = (1, 0, -1)$ $\quad u_k = r_k - y_k$ $\quad y(t) = 2\int_0^t u(\tau)\,d\tau$

$T = 1$ $\quad$ D/A converter : Zero-order hold

$y(0) = 0$

$y_0 = 0$

$0 \le t < 1$

$r_0 = 1$ $\quad u_0 = r_0 - y_0 = 1 - 0$ (3) $\quad \boxed{u(t) = \sigma(t)}$ or $\boxed{u(t) = 1 \quad 0 \le t < 1}$ (1)

$y(t) = 2\int_0^t u(\tau)\,d\tau = 2\int_0^t 1\,d\tau = 2t$ (3) $\quad y(T) = y(1) = 2 = y_1$

$1 \le t < 2$

$r_1 = 0 \quad y_1 = 2 \quad u_1 = r_1 - y_1 = 0 - 2 = -2$ (3) $\quad \boxed{u(t) = -2\sigma(t-1)}$ or $\boxed{u(t) = -2 \quad 1 \le t < 2}$ (1)

$$y(t) = 2\int_0^t u(\tau)\,d\tau = 2\left(\int_0^1 u(\tau)\,d\tau + \int_1^t u(\tau)\,d\tau\right]$$

$$= 2\left[\int_0^1 1\,d\tau + \int_1^t (-2)\,d\tau\right]$$

$$= 2\left[\tau\big|_0^1 + (-2)\,\tau\big|_1^t\right]$$

$$= 2\left[1 - 2(t-1)\right] = 2\left[3 - 2t\right]$$

$y(t) = 6 - 4t$ (3)

$y(2) = -2 \quad y_2 = -2$

$2 \le t < 3$

$r_2 = -1 \quad y_2 = -2 \quad u_2 = r_2 - y_2 = -1 - (-2) = 1$ (3) $\Rightarrow$ $\boxed{u(t) = 1\sigma(t-2)}$ or $\boxed{u(t) = 1 \quad 2 \le t < 3}$ (1)

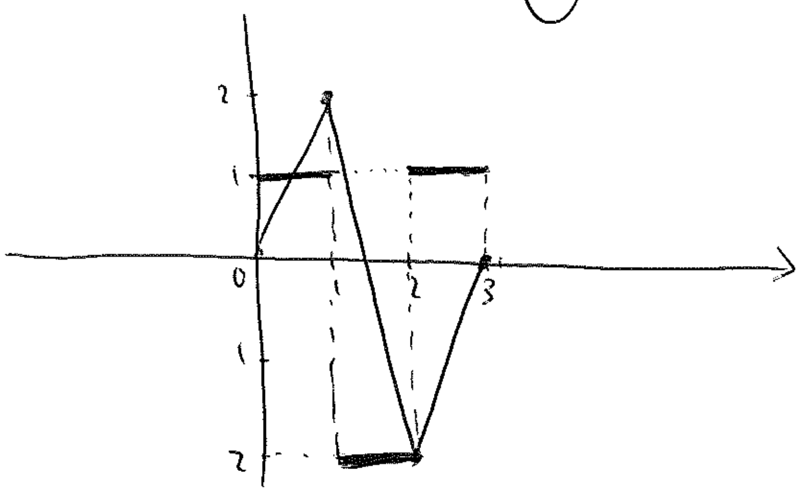
$$y(t) = 2\int_0^t u(\tau)\,d\tau = 2\left(\int_0^1 \underbrace{u(\tau)}_{1}\,d\tau + \int_1^2 \underbrace{u(\tau)}_{-2}\,d\tau + \int_2^t \underbrace{u(\tau)}_{1}\,d\tau\right]$$

Q2 continue

$$y(t) = 2\left[\tau\Big|_0^1 - 2\tau\Big|_1^2 + \tau\Big|_2^t\right]$$

$$= 2\left[1 - 2 + (t-2)\right] = 2[t-3] \quad (3)$$

(4)



Q3

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad y = [0 \quad 1]\, x_k \quad , \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_k = (1, -1, 1, -1, \ldots) \xrightarrow{Z} U(z) = \frac{z}{z+1}$$

(a) $X(z) = \left[zI - A_d\right]^{-1} b_d\, U(z)$

$$zI - A_d = \begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix}$$

$$= \frac{\begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix}}{z^2+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{z}{z+1}$$

7

$$X(z) = \begin{bmatrix} \dfrac{z}{(z+1)(z^2+1)} \\[2ex] \dfrac{z^2}{(z+1)(z^2+1)} \end{bmatrix}$$

$$\frac{z^2}{(z+1)(z^2+1)} = \frac{A}{z+1} + \frac{B}{z-j} + \frac{C}{z+j}$$

$$\cancel{z}^2 = A(z^2+1) + B(z+1)(z+j) + C(z+1)(z-j)$$

put $z = -1 \Rightarrow$ $\quad A = \frac{1}{2}$

put $z = j \Rightarrow \quad -1 = B(j+1)2j \quad B = \frac{-1}{2j(j+1)}$

$$B = \frac{j}{2(j+1)} = \frac{j(1-j)}{2 \times 2} = \frac{1+j}{4}$$

13

$$C = \frac{1-j}{4}$$

$$\frac{z^2}{(z+1)(z^2+1)} = \frac{1}{2}\frac{1}{z+1} + \frac{1+j}{4}\frac{1}{z-j} + \frac{1-j}{4}\frac{1}{z+j}$$

2

$$X(z) = \begin{bmatrix} z^{-1}\left[\dfrac{1}{z+1} + \dfrac{1-j}{4}\dfrac{1}{z-j} + \dfrac{1+j}{4}\dfrac{1}{z+j}\right] \\[2ex] \left[\dfrac{1}{z+1} + \dfrac{1-j}{4}\dfrac{1}{z-j} + \dfrac{1+j}{4}\dfrac{1}{z+j}\right] \end{bmatrix}$$

$$x_k = \begin{bmatrix} \frac{1}{2}(-1)^{k-2}\sigma_{k-2} + \frac{1+j}{4}(j)^{k-2}\sigma_{k-2} + \frac{1-j}{4}(-j)^{k-2}\sigma_{k-2} \\[1ex] (-1)^{k-1}\sigma_{k-1}\frac{1}{2} + \frac{1+j}{4}(j)^{k-1}\sigma_{k-1} + \frac{1-j}{4}(-j)^{k-1}\sigma_{k-1} \end{bmatrix}$$

6

2

$$y_k = \frac{1}{2}(-1)^{k-1}\sigma_{k-1} + \frac{1+j}{4}(j)^{k-1}\sigma_{k-1} + \frac{1-j}{4}(-j)^{k-1}\sigma_{k-1}$$

$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y_0 = 0$, $x_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $y_1 = 1$, $x_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow y_2 = -1$

Q4 $G(s) = \frac{1}{s}$ $F(s) = \frac{G(s)}{s} = \frac{1}{s^2} \xrightarrow{\mathcal{L}^{-1}} f(t) = t$ $f(t)\Big|_{t=kT} = kT$

$F(z) = \frac{Tz}{(z-1)^2}$ $G(z) = \frac{z-1}{z} F(z) = \frac{T}{z-1}$

$$G(z) = \frac{T}{z-1}$$

Q5 $A_d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $zI - A_d = \begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix}$ $\det(zI - A_d) = z^2 + 1$

$z_1 = j$

$z_2 = -j$

(a) $z_1 = j$ and $z_2 = -j$ are over unit circle

NOT ASYMPTOTICALLY STABLE

(b) At least $z_1 = j$ and $z_2 = -j$ are over the unit circle

STABLE

(c) For BIBO stability roots of $(zI - A_d)$ should all be inside unit circle.

NOT BIBO stable

Q8 $$\frac{U(z)}{E(z)} = K_p \frac{(\gamma T_i + 1)z + (1 - \gamma T_i)}{\gamma T_i (z-1)}$$

$$U(z)\,\gamma T_i \left[1 - \frac{1}{z}\right] = E(z)\, K_p \left[(\gamma T_i + 1) + \frac{(1 - \gamma T_i)}{z}\right]$$

$\Big\downarrow z^{-1}$

$$\boxed{u_k = u_{k-1} + \frac{K_p(\gamma T_i + 1)}{\gamma T_i} e_k + \frac{K_p(1 - \gamma T_i)}{\gamma T_i} e_{k-1}}$$