30-03-2017

MECE480 MIDTERM

Q1) A discrete time system state equation is given by the formula

$$\dot{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \quad y = [0 \quad 1]\cancel{y}\, x_k$$

Assuming that initial condition is $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $u_k=(1)^k=(1,1,1,1,....)$ find x_k and y_k using Z-transformation.

(30 point)

Q2) A digital Control system is given with the follwing properties

Plant description: $y(t) = y(0) + \int_0^t u(\tau)d\tau$

Reference sequence: $r_k=(0, 1, 0)$

Control algorithm: $u_k=r_k-y_k$

Samplin time: T=1

D/A converter: zero-order hold

Assuming that y(0)=1, find u(t) and y(t) when $0 \le t \le 3$ seconds and drow them. **(30 point)**

Q3) A discrete time system state equation is given by the formula

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \quad y_k = [1 \quad 0]x_k$$

a) Explain why this system is not BIBO (bounded input bounded output). **(5 point)**
b) Explain why this system is not aysmptotically stable. **(5 point)**
c) Show that the bounded input sequence $u_k = \sin(k\frac{\pi}{2})\sigma_k$ makes the output unbounded. **(20 point)**

Q4) A continuous time transfer fucntion si given by the formula $G(s) = \frac{1}{s}$. Find the discrete time trasnfer function G(z). **(10 point)**

Q1

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$$x_{k+1} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A_d} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad y = [0 \quad 1] x_k$$

$$zI - A_d = \begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix} \qquad X(z) = (zI - A_d)^{-1} b_d U(z)$$

$$(zI - A_d)^{-1} = \frac{\begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix}}{z^2+1}$$

$$u_k = [1, 1, \ldots] \xrightarrow{Z} U(z) = \frac{z}{z-1}$$

$$X(z) = [zI - A_d]^{-1} b_d U(z) = \frac{\begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix}}{z^2+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{z}{z-1} = \frac{z}{(z-1)(z^2+1)} \begin{bmatrix} 1 \\ z \end{bmatrix}$$

$$X(z) = \begin{bmatrix} \dfrac{z}{(z-1)(z^2+1)} \\[2ex] \dfrac{z^2}{(z-1)(z^2+1)} \end{bmatrix}$$

10 point

$$\frac{z}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{B}{z-j} + \frac{B^*}{z+j}$$

10 point

$$A = \left. \frac{z(z-1)}{(z-1)(z^2+1)} \right|_{z=1} = \left. \frac{z}{z^2+1} \right|_{z=1} = \frac{1}{2}$$

$$B = \left. \frac{z(z-j)}{(z-1)(z-j)(z+j)} \right|_{z=j} = \left. \frac{z}{(z-1)(z+j)} \right|_{z=j} = \frac{j}{2j[j-1]} = \frac{1}{2[j-1]} = \frac{-1}{2(1-j)}\frac{(1+j)}{(1+j)} = \frac{-[1+j]}{2\times 2}$$

$$B = \frac{-1-j}{4} \qquad B^* = \frac{-1+j}{4}$$

$$\frac{z}{(z-1)(z^2+1)} = \frac{1}{2}\,\frac{1}{z-1} + \left(\frac{-1-j}{4}\right)\frac{1}{z-j} + \left(\frac{-1+j}{4}\right)\frac{1}{z+j}$$

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$\downarrow$

$$= \frac{1}{2}\,\frac{z^{-1}}{1-z^{-1}} + \left(\frac{-1-j}{4}\right)\frac{z^{-1}}{1-jz^{-1}} + \left(\frac{-1+j}{4}\right)\frac{z^{-1}}{1+jz^{-1}}$$

$$= \frac{1}{2}\,\sigma_{k-1} + \left(\frac{-1-j}{4}\right) e^{j(k-1)}\sigma_{k-1} + \left(\frac{-1+j}{4}\right) e^{-j(k-1)}\sigma_{k-1}$$

$$= \frac{1}{2}\sigma_{k-1} + \left(-\frac{1}{4}\right)\left[\underline{e^{j(k-1)}\sigma_{k-1} + e^{-j(k-1)}}\right] + \frac{-1}{4}\left[je^{j(k-1)} - je^{-j(k-1)}\right]\sigma_{k-1}$$

$$= \frac{1}{2}\sigma_{k-1} + \left(-\frac{1}{2}\right)\left(\frac{e^{j(k-1)} + e^{-j(k-1)}}{2}\right)\sigma_{k-1} + -\frac{1}{4}j(2j)\left[\frac{e^{j(k-1)} - e^{-j(k-1)}}{2j}\right]\sigma_{k-1}$$

$$j = e^{j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)$$

$$(-j) = e^{-j\frac{\pi}{2}}$$

$$= \frac{1}{2}\sigma_{k-1} + \left(-\frac{1}{2}\right)\cos$$

$$= \frac{1}{2}\,\cancel{\frac{1}{z-1}}\,\sigma_{k-1} + \left(\frac{-1-j}{4}\right) j^{(k-1)}\sigma_{k-1} + \left(\frac{-1+j}{4}\right)(-j)^{(k-1)}\sigma_{k-1}$$

$$= \frac{1}{2}\sigma_{k-1} + \left(-\frac{1}{4}\right)\sigma_{k-1}\quad j^{(k-1)} + (-j)^{(k-1)}$$

σ point

$$= \frac{1}{2}\sigma_{k-1} + \left(-\frac{1}{4}\right)\sigma_{k-1}\, e^{j\frac{\pi}{2}(k-1)} + \left(-\frac{1}{4}\right)e^{-j\frac{\pi}{2}(k-1)}$$

$$+ \left(-\frac{1}{4}\right)\sigma_{k-1}\left(\frac{je^{j\frac{\pi}{2}(k-1)} - je^{-j\frac{\pi}{2}(k-1)}}{2j}\right)2j$$

$$= \frac{1}{2}\sigma_{k-1} + \left(-\frac{1}{4}\right)2\left[\frac{e^{j\frac{\pi}{2}(k-1)} + e^{-j(\frac{\pi}{2})(k-1)}}{2}\right] + \left(-\frac{1}{4}j\right)2j\left(\frac{e^{j\frac{\pi}{2}(k-1)} - e^{-j\frac{\pi}{2}(k-1)}}{2j}\right.$$

$$= \frac{1}{2}\sigma_{k-1} - \frac{1}{2}\cos\left((k-1)\frac{\pi}{2}\right)\sigma_{k-1} + \frac{1}{2}\sin\left((k-1)\frac{\pi}{2}\right)\sigma_{k-1}$$

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$$X(z) = \begin{bmatrix} \dfrac{z}{(z-1)(z^2+1)} \\ \\ z\,\dfrac{z}{(z-1)(z^2+1)} \end{bmatrix} \xrightarrow{z^{-1}} x_k = \begin{bmatrix} \frac{1}{2}\sigma_{k-1} - \frac{1}{2}\cos\left((k-1)\frac{\pi}{2}\right)\sigma_{k-1} + \frac{1}{2}\sin\left((k-1)\frac{\pi}{2}\right)\sigma_{k-1} \\ \\ \frac{1}{2}\sigma_k - \frac{1}{2}\cos\left(k\frac{\pi}{2}\right)\sigma_k + \frac{1}{2}\sin\left(k\frac{\pi}{2}\right)\sigma_k \end{bmatrix}$$

2 point

$$y_k = [0 \ \ 1]\,x_k = \frac{1}{2}\sigma_k - \frac{1}{2}\cos\left(k\frac{\pi}{2}\right)\sigma_k + \frac{1}{2}\sin\left(k\frac{\pi}{2}\right)\sigma_k$$

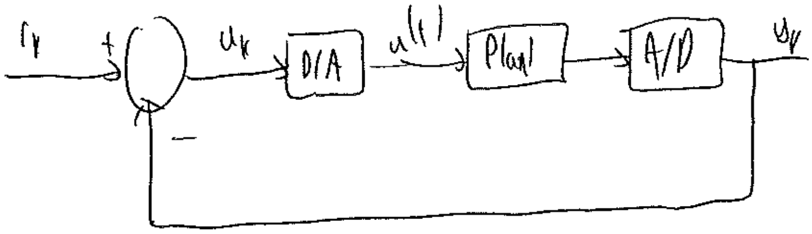
2 point

(Q2)

$y(t) = \int_0^t u(\tau)\, d\tau \qquad r_k = [0, +1, 0]$

Control algorithm $u_k = (r_k - y_k)$ sampling time $T = 1$

D/A convertor : zero-order hold



$y(0) = 1$

$0 \le t < 1$

$r_0 = 0, \; y_0 = 1 \Rightarrow u_0 = (r_0 - y_0) = (0-1) = -1$

hence $u(t) = -1\sigma(t) \quad 0 \le t < 1$

$y(t) = y(0) + \int_0^t u(\tau)d\tau = 1 + \int_0^t (-1)d\tau = 1 - \tau\Big|_0^t = 1 - t$ (8)

$y(0) = 1 \quad y(1) = 0$

$1 \le t < 2$

$r_1 = 1 \quad y_1 = 0 \Rightarrow u_1 = (r_1 - y_1) = (1-0) = 1$

hence $u(t) = 1\sigma(t-1) \quad 1 \le t < 2$

$y(t) = y(0) + \int_0^t u(\tau)d\tau = 1 + \int_0^1 (-1)d\tau + \int_1^t 1\, d\tau$ (8)

$= 1 - \tau\Big|_0^1 + \tau\Big|_1^t = 1 - 1 + (t-1) = (t-1)$

$y(1) = 0 \quad y(2) = 1 \quad y_2 = 1$

$2 \le t < 3$

$r_2 = 0 \quad y_2 = 1 \Rightarrow u_2 = r_2 - y_2 = (0-1) = -1$

hence $u(t) = -1\sigma(t-2) \quad 2 \le t < 3$

$y(t) = y(0) + \int_0^t u(\tau)d\tau = 1 + \int_0^1 (-1)d\tau + \int_1^2 1\, d\tau + \int_2^t (-1)d\tau$ (8)

$= 1 - \tau\Big|_0^1 + \tau\Big|_1^2 - \tau\Big|_2^t$

$= 1 - 1 + (2-1) - (t-2) = 1 - (t-2) = 3 - t$

$y(2) = 1 \qquad y(3) = 0$

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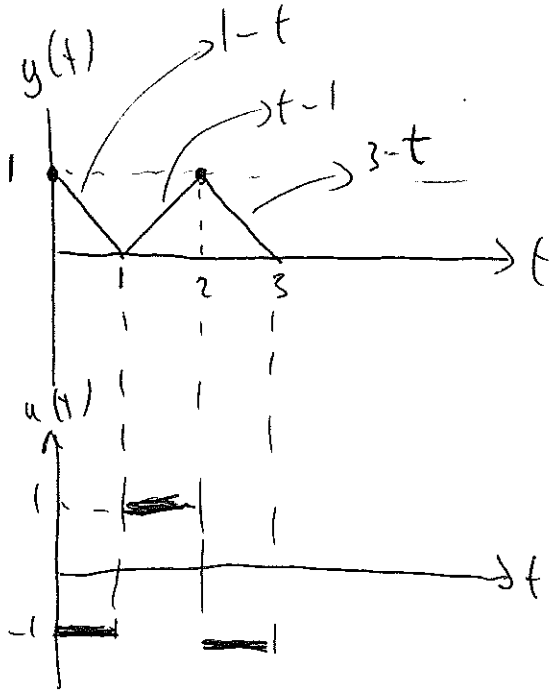

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Q3 →

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(a) not BIBO stable (S)

eigenvalues are over unit circle $[p_1 = j,\ p_2 = -j]$

(b) not asymptotically stable (eigenvalues are over unit circle)

$[p_1 = j,\ p_2 = -j]$ (S)

(c)

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad y_k = [1 \ \ 0]\, x_k$$

$$z X(z) - x_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X(z) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(z) \qquad Y(z) = [1 \ \ 0]\, X(z)$$

$x_0 \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → initial condition

$$\left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right] X(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(z)$$

$$\begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix} X(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(z)$$

(S)

$$X(z) = \frac{\begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix}}{z^2+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(z)$$

$$X(z) = \begin{bmatrix} \frac{1}{z^2+1} \\ \frac{z}{z^2+1} \end{bmatrix} U(z) \qquad Y(z) = [1 \ \ 0]\, X(z)$$

$$Y(z) = \frac{1}{z^2+1} U(z)$$

$$k x_k \xrightarrow{z} -z \frac{dX(z)}{dz}$$

$$k(j)^k \sigma_k \xrightarrow{z} -z \frac{(-j)}{(z-j)^2}$$

$$k(-j)^k \sigma_k \xrightarrow{z} \frac{-z(j)}{[z+j]^2}$$

$$u_k = \sin\left(k\frac{\pi}{2}\right)\sigma_k \Longrightarrow U(z) = \frac{z}{z^2+1}$$

$$Y(z) = \frac{1}{z^2+1}U(z) = \frac{1}{z^2+1} \times \frac{z}{z^2+1} = \frac{z}{(z^2+1)^2}$$

$$Y(z) = \frac{z}{(z^2+1)^2} = \frac{A}{z+j} + \frac{B}{(z+j)^2} + \frac{C}{(z-j)} + \frac{D}{(z-j)^2}$$ 3

$$z = A(z+j)(z-j)^2 + B(z-j)^2 + C(z-j)(z+j)^2 + D(z+j)^2$$

$$z = j \Rightarrow \quad j = D(2j)^2 \quad j = -4D \quad D = \frac{-j}{4}$$

$$z = -j \Rightarrow \quad -j = B(-2j)^2 \quad -j = -4B \quad B = \frac{j}{4}$$

A and C will be computed to be 0

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$$Y(z) = \frac{\frac{j}{4}}{(z+j)^2} + \frac{\frac{-j}{4}}{(z-j)^2}$$

$$= \frac{1}{4}\left[z^{-1}\frac{jz}{(z+j)^2} + z^{-1}\frac{(-j)z}{(z-j)^2}\right]$$

$$Y(z) = -\frac{1}{4}\left[\underbrace{z^{-1}\frac{-jz}{(z+j)^2}}_{} + z^{-1}\frac{jz}{(z-j)^2}\right]$$

$$\Big\downarrow Z^{-1}$$

$$y_k = -\frac{1}{4}\left[(k-1)(-j)^{k-1}\sigma_{k-1} + (k-1)(j)^{k-1}\sigma_{k-1}\right] = -\frac{1}{4}(k-1)(-j)^{k-1}\sigma_{k-1} - \frac{1}{4}(k-1)(j)^{k-1}\sigma_{k-1}$$

2

$$y_k = -\frac{1}{4}\left[(k-1)\sigma_{k-1}\right]\left[(-j)^{k-1} + (j)^{k-1}\right]$$

$$= -\frac{1}{4}\left[(k-1)\sigma_{k-1}\right]2\left[\frac{e^{\left(-j\frac{\pi}{2}\right)(k-1)} + e^{\left(j\frac{\pi}{2}\right)(k-1)}}{2}\right]$$ 2

$$= -\frac{1}{2}(k-1)\sigma_{k-1}\cos\left[(k-1)\frac{\pi}{2}\right] = -\frac{1}{2}(k-1)\cos\left[(k-1)\frac{\pi}{2}\right]\sigma_{k-1}$$

Q4 $G(s) = \frac{1}{s}$ $F(s) = \frac{G(s)}{s} = \frac{1}{s^2} \xrightarrow{\mathcal{L}^{-1}} f(t) = t$

$f(t)\Big|_{t=kT} = kT \xrightarrow{z} F(z) = \frac{Tz}{(z-1)^2}$ $G(z) = \frac{z-1}{z} F(z) = \frac{T}{z-1}$

$$G(z) = \frac{T}{z-1}$$

$Y(z) = \frac{-1}{(z^2+1)} U(z)$ $\qquad [z^2+1]\, Y(z) = -U(z)$

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$z^2 Y(z) + Y(z) = -U(z)$ $\qquad u_k = \sin\left(k\frac{\pi}{2}\right)$

$y_{k+2} + y_k = -u_k = -\sin\left(k\frac{\pi}{2}\right)$

$y_k\text{-particular} = M_1 k \sin\left(k\frac{\pi}{2}\right) + M_2 k \cos\left(k\frac{\pi}{2}\right)$

$y_{k+2} = M_1 (k+2) \sin\left((k+2)\frac{\pi}{2}\right) + M_2 (k+2) \cos\left((k+2)\frac{\pi}{2}\right) =$

$M_1 (k+2) \sin\left((k+2)\frac{\pi}{2}\right) + M_2 (k+2) \cos\left((k+2)\frac{\pi}{2}\right) = -\sin\left(k\frac{\pi}{2}\right)$

$y_{k+2} = -M_1 (k+2) \sin\left(k\frac{\pi}{2}\right) - M_2 (k+2) \cos\left(k\frac{\pi}{2}\right)$

$y_{k+2} + y_k = -\sin\left(k\frac{\pi}{2}\right)$

$-2M_1 \sin\left(k\frac{\pi}{2}\right) - 2M_2 \cos\left(k\frac{\pi}{2}\right) = -\sin\left(k\frac{\pi}{2}\right)$

$M_2 = 0 \quad M_1 = \frac{1}{2}$

$y_k\text{-particular} = \frac{1}{2} k \sin\left(k\frac{\pi}{2}\right)$

$y_k = \frac{1}{2}(k-1) \sin\left(k\frac{\pi}{2}\right) \sigma_{k-1}$ $\qquad y_0 = 0 \quad y_1 = 0$

$y_2 =$