Ex: The difference equation is given by the formula

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$$y(k) + y(k-2) = \sin\left(k\frac{\pi}{2}\right)$$ with initial conditions $y(0)=1$ $y(1)=6$

Step 1: Find the homogeneous solution

$y(k) + y(k-2) = 0 \Rightarrow y_h(k) = A\zeta^k$ (general solution homogeneous),

$y_h(k) = A\zeta^{k-2}$

$$A\zeta^k + A\zeta^{k-2} = 0 \qquad \zeta^k + \zeta^{k-2} = 0 \qquad \zeta^{k-2}(\zeta^2+1) = 0$$

$$\zeta_{1,2} = \mp j \qquad j = e^{j90^\circ} = e^{j\frac{\pi}{2}} \qquad -j = e^{-j90} = e^{-j\frac{\pi}{2}}$$

$$y_h(k) = Ae^{j\frac{\pi}{2}k} + Be^{-j\frac{\pi}{2}k}$$

or by transformation

$$y_h(k) = K_1\cos\left(k\frac{\pi}{2}\right) + K_2\sin\left(k\frac{\pi}{2}\right)$$

Step 2: Find the particular solution

$$k\frac{\pi}{2} = k\omega \qquad \omega = \frac{\pi}{2} \qquad \omega = 2\pi f \qquad \frac{\pi}{2} = 2\pi f \qquad f = \frac{1}{4} \qquad T = 4 \text{ period}$$

$$e^{j\frac{\pi}{2}} = e^{j\omega} = j = \zeta_1$$

(hence we have to increase the order of the particular solution

$$y_p(k) = C_1 k\cos\left(k\frac{\pi}{2}\right) + D_1 k\sin\left(k\frac{\pi}{2}\right)$$

put this solution in the diff equation $\to y_p(k)$

$$y(k)+y(k-2)=\sin\left(k\frac{\pi}{2}\right)$$

$$C_1 k\cos\left(k\frac{\pi}{2}\right)+D_1 k\sin\left(k\frac{\pi}{2}\right)+C_1(k-2)\cos\left((k-2)\frac{\pi}{2}\right)$$
$$+D_1(k-2)\sin\left((k-2)\frac{\pi}{2}\right)=\sin\left(k\frac{\pi}{2}\right)$$

$$C_1 k\cos\left(k\frac{\pi}{2}\right)+D_1 k\sin\left(k\frac{\pi}{2}\right)+C_1(k-2)\cos\left(\frac{k\pi}{2}-\pi\right)$$
$$+D_1(k-2)\sin\left(\frac{k\pi}{2}-\pi\right)=\sin\left(k\frac{\pi}{2}\right)$$
$$C_1 k\cos\left(k\frac{\pi}{2}\right)+D_1 k\sin\left(k\frac{\pi}{2}\right)-C_1(k-2)\cos\left(\frac{k\pi}{2}\right)$$
$$-D_1(k-2)\sin\left(\frac{k\pi}{2}\right)=\sin\left(k\frac{\pi}{2}\right)$$

$$2C_1\cos\left(\frac{k\pi}{2}\right)+2D_1\sin\left(\frac{k\pi}{2}\right)=\sin\left(k\frac{\pi}{2}\right)$$

$$C_1=0 \qquad D_1=\frac{1}{2}$$

$$y_p(k)=\frac{1}{2}k\sin\left(k\frac{\pi}{2}\right)$$

step 3: write total solution

$$y(k)=y_h(k)+y_p(k)$$
$$y(k)=K_1\cos\left(\frac{k\pi}{2}\right)+K_2\sin\left(\frac{k\pi}{2}\right)+\frac{1}{2}k\sin\left(k\frac{\pi}{2}\right)$$
$$y(0)=1 \qquad y(0)=K_1=1 \quad K_1=1$$

$y(1)=0 \qquad y(1)=0=K_1\cos\left(\frac{\pi}{2}\right)+K_2\sin\left(\frac{\pi}{2}\right)+\frac{1}{2}\times1\times\sin\left(\frac{\pi}{2}\right)$

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$0=K_2+\frac{1}{2} \qquad K_2=-\frac{1}{2}$ //

$y(k)=1\cos\left(\frac{k\pi}{2}\right)+\left(-\frac{1}{2}\right)\sin\left(\frac{k\pi}{2}\right)+\frac{1}{2}k\sin\left(k\frac{\pi}{2}\right)$ //

Ex: $y(k)+y(k-2)=\sin(k\pi)=u(k) \quad y(0)=1, \; y(1)=0$

Step 1: Homogeneous solution

$$y(k)+y(k-2)=0 \Rightarrow y_h(k)=A\cos\left(k\frac{\pi}{2}\right)+B\sin\left(k\frac{\pi}{2}\right)$$

$$\xi_{1,2}=\mp j \Rightarrow \begin{array}{l} j=e^{j\frac{\pi}{2}} \\ -j=e^{-j\frac{\pi}{2}} \end{array}$$

↳ natural frequencies

Step 2 Particular solution

$$u(k)=\sin(k\pi)=\sin(kw) \Rightarrow k\pi=kw \quad w=\pi$$

$$e^{jw}=e^{j\pi}=\cos(\pi)+j\sin(\pi)=-1 \neq \xi_1 \text{ or } \xi_2$$

hence the particular solution is

$$y_p(k)=C_1\cos(k\pi)+D_1\sin(k\pi)$$

put the solution in the diff equation $y_p(k)$

$$y(k)+y(k-2)=\sin(k\pi)$$

$$C_1\cos(k\pi)+D_1\sin(k\pi)+C_1\cos((k-2)\pi)+D_1\sin((k-2)\pi)=\sin(k\pi)$$

$$C_1\cos(k\pi)+D_1\sin(k\pi)+C_1\cos(k\pi)+D_1\sin(k\pi)=\sin(k\pi)$$

$$2C_1\cos(k\pi)+2D_1\sin(k\pi)=\sin(k\pi) \qquad C_1=0 \quad D_1=\frac{1}{2}$$

$$y_p(k)=\frac{1}{2}\sin(k\pi)$$

Step 3: Total solution

$$y(k)=y_p(k)+y_h(k)=\frac{1}{2}\sin(k\pi)+A\cos\left(k\frac{\pi}{2}\right)+B\sin\left(k\frac{\pi}{2}\right)$$

$$y(0)=1 = \frac{1}{2}\sin(0) + A\cos(0) + B\sin(0) \qquad A=1$$

$$y(1) = \frac{1}{2}\sin(\pi) + A\cos\left(\frac{\pi}{2}\right) + B\sin\left(\frac{\pi}{2}\right) = 0$$

$$0 + 0 + B = 0 \qquad B=0$$

$$y(k) = \frac{1}{2}\sin(k\pi) + \cos\left(k\frac{\pi}{2}\right)$$

Ex! $x(t) = \begin{cases} \sin(\omega t) & , t \geq 0 \\ 0 & , t < 0 \end{cases}$ $\quad \sin(\omega t) = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$

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$$x(kT) = x_k = \sin(k\omega T) = \frac{1}{2j}\left[e^{jk\omega T} - e^{-jk\omega T}\right]$$

$$z\left\{\sin(k\omega T)\right\} = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}z^{-1}} - \frac{1}{1-e^{-j\omega T}z^{-1}}\right]$$

$$= \frac{z\sin(\omega T)}{z^2 - 2z\cos(\omega T) + 1}$$

Ex! $x(t) = \begin{cases} \cos(\omega t) & , t \geq 0 \\ 0 & , t < 0 \end{cases}$ $\quad x_k = x(kT) = \cos(k\omega T) \quad k = 0, 1, \ldots$

$$\cos(k\omega T) = \frac{1}{2}\left[e^{jk\omega T} + e^{-jk\omega T}\right]$$

$$z\left\{\cos(k\omega T)\right\} = \frac{1}{2}\left[\frac{1}{1-e^{j\omega T}z^{-1}} + \frac{1}{1-e^{-j\omega T}z^{-1}}\right]$$

$$= \frac{z^2 - z\cos(\omega T)}{z^2 - 2z\cos(\omega T) + 1}$$

Ex $X(z) = \dfrac{z^2+z+2}{(z-1)(z^2-z+1)} = \dfrac{4}{z-1} + \dfrac{-3z+2}{z^2-z+1}$ (using partial fraction expansion)

$$= \frac{4z^{-1}}{1-z^{-1}} + \frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}$$

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$$Z\{e^{-akT}\cos(\omega kT)\} = \frac{1-e^{-aT}z^{-1}\sin(\omega T)}{1-2e^{-aT}z^{-1}\cos(\omega T)+e^{-2aT}z^{-2}}$$

$$Z\{e^{-akT}\sin(\omega kT)\} = \frac{e^{-aT}z^{-1}\sin(\omega T)}{1-2e^{-aT}z^{-1}\cos(\omega T)+e^{-2aT}z^{-2}}$$

$$1-z^{-1}+z^{-2} = 1-2e^{-aT}z^{-1}\cos(\omega T)+e^{-2aT}z^{-2}$$

$$-1 = -2e^{-aT}\cos(\omega T) \rightarrow e^{-aT} = 1 \Rightarrow a = 0$$

$$\rightarrow \cos(\omega T) = \frac{1}{2} \qquad \omega T = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\sin(\omega T) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow X(z) = \frac{4z^{-1}}{1-z^{-1}} + \frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}$$

$$= \frac{4z^{-1}}{1-z^{-1}} + \frac{-3z^{-1}+1.5z^{-2}+0.5z^{-2}}{1-z^{-1}+z^{-2}}$$

$$X(z) = \frac{4z^{-1}}{1-z^{-1}} + \frac{-3z^{-1}+1.5z^{-2}}{1-z^{-1}+z^{-2}} + \frac{0.5z^{-2}}{1-z^{-1}+z^{-2}}$$

$$= \frac{4z^{-1}}{1-z^{-1}} + (-3)\left[\frac{z^{-1}-0.5z^{-2}}{1-z^{-1}+z^{-2}}\right] + \frac{0.5}{\frac{\sqrt{3}}{2}}\frac{\frac{\sqrt{3}}{2}z^{-2}}{1-z^{-1}+z^{-2}} \quad //$$

$$X(z) = \frac{4z^{-1}}{1-z^{-1}} + (-3)\left[\frac{z^{-1}-0.5z^{-2}}{1-z^{-1}+z^{-2}}\right] + \frac{1}{\sqrt{3}}\ \frac{\frac{\sqrt{3}}{2}z^{-2}}{1-z^{-1}+z^{-2}}$$

$$\downarrow$$

$$x(k) = 6\,(k-1) - 3\cos\frac{(k-1)\pi}{3}\,\sigma(k-1) + \frac{1}{\sqrt{3}}\sin\frac{(k-1)\pi}{3}\,\sigma(k-1)$$