(1)

MECE 480

Extra ~~wetting~~ question

DELAYED COMPUTATION

[Chapter 10]

$$\dot{x} = \overbrace{\begin{bmatrix} -100 & 0 \\ 0 & -100 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{b} u \qquad y = \underbrace{[0 \;\; 1]}_{C^T} x$$

$\Delta = 0.005$

$T = 0.01$

$$(sI - A) = \begin{pmatrix} s+100 & 0 \\ 0 & s+100 \end{pmatrix}$$

application of u_k is delayed by $\Delta = 0.005$ seconds

$\Delta < T$

$$A_d = e^{AT}$$

$$e^{At} = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\} = \mathcal{L}^{-1}\left\{ \frac{\begin{pmatrix} s+100 & 0 \\ 0 & s+100 \end{pmatrix}}{(s+100)^2} \right\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s+100} & 0 \\ 0 & \frac{1}{s+100} \end{bmatrix} \right\}$$

(a) obtain

$$e^{At} = \begin{bmatrix} e^{-100t} & 0 \\ 0 & e^{-100t} \end{bmatrix} \qquad A_d = e^{AT} = \begin{bmatrix} e^{-100 \times 0.01} & 0 \\ 0 & e^{-100 \times 0.01} \end{bmatrix} = \begin{bmatrix} \frac{1}{e} & 0 \\ 0 & \frac{1}{e} \end{bmatrix}$$

$$\tilde{b}_d = \int_0^{\Delta} e^{A\tau} b \, d\tau = \int_0^{\Delta = 0.005} \begin{bmatrix} e^{-100\tau} & 0 \\ 0 & e^{-100\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^{0.005} \begin{bmatrix} 0 \\ e^{-100\tau} \end{bmatrix} d\tau = \begin{bmatrix} 0 \\ -\frac{e^{-100\tau}}{100} \end{bmatrix} \Bigg|_0^{0.005}$$

$$\tilde{b}_d = \begin{bmatrix} 0 \\ -\frac{1}{100}\left[e^{-0.5} - 1\right] \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{100}\left[1 - \frac{1}{\sqrt{e}}\right] \end{bmatrix}$$

$$b_d = \int_{\Delta}^{T} e^{A\tau} b \, d\tau = \int_{0.005}^{0.01} \begin{bmatrix} 0 \\ +e^{-100\tau} \end{bmatrix} d\tau = \left\{ \begin{bmatrix} 0 \\ -\frac{e^{-100\tau}}{100} \end{bmatrix} \right\} \Bigg|_{0.005}^{1}$$

$$b_d = \begin{bmatrix} 0 \\ -\frac{1}{100}\left(\frac{1}{e} - \frac{1}{\sqrt{e}}\right) \end{bmatrix} = \frac{0}{\frac{1}{100}\left[\frac{1}{\sqrt{e}} - \frac{1}{e}\right]}$$

$$y_1 = C_d^T x_k$$
$$= C$$
$$C = C_d^T = (0 \; 1)$$

then

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A_d & \tilde{b}_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} + \begin{bmatrix} b_d \\ 1 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} C_d^T & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}$$

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{e} & 0 & 0 \\ 0 & \frac{1}{e} & \frac{1}{100}\left[1 - \frac{1}{\sqrt{e}}\right] \\ 0 & 0 & 0 \end{bmatrix}}_{A_{new}} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{100}\left[\frac{1}{\sqrt{e}} - \frac{1}{e}\right] \\ 1 \end{bmatrix}}_{b_{new}} u_k$$

Delayed system state space model

$$y_k = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C_{new}^T} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}$$

(b) Find eigenvalues of A_{new}

$$(zI - A_{new}) = \begin{bmatrix} z - \frac{1}{e} & 0 & 0 \\ 0 & z - \frac{1}{e} & -\frac{1}{100}\left[1 - \frac{1}{\sqrt{e}}\right] \\ 0 & 0 & z \end{bmatrix}$$

(b) $\det[zI - A_{new}] = \left[z - \frac{1}{e}\right](-1)^{1+1}\left[\left[z-\frac{1}{e}\right]z - \left(\frac{-1}{100}\left(1-\frac{1}{\sqrt{e}}\right)\right)\times 0\right]$

$= \left[z - \frac{1}{e}\right]^2 z$

$z_1 = 0 \qquad z_{2,3} = \frac{1}{e}$ $\Rightarrow$ eigenvalues of A_d

(c) Find the discrete system transfer function

$H(z) = C_{new}^T\left[zI - A_{new}\right]^{-1} b_{new}$

$$= [0\ 1\ 0]\begin{bmatrix} z-\frac{1}{e} & 0 & 0 \\ 0 & z-\frac{1}{e} & \frac{-1}{100}\left(1-\frac{1}{\sqrt{e}}\right) \\ 0 & 0 & z \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{100}\left[\frac{1}{\sqrt{e}} - \frac{1}{e}\right] \\ 1 \end{bmatrix}$$

$$= [0\ 1\ 0]\frac{\begin{bmatrix} \begin{vmatrix} z-\frac{1}{e} & \frac{-1}{100}\left(1-\frac{1}{\sqrt{e}}\right) \\ 0 & z \end{vmatrix} & -\begin{vmatrix} 0 & \frac{-1}{100}\left(1-\frac{1}{\sqrt{e}}\right) \\ 0 & z \end{vmatrix} & \begin{vmatrix} 0 & z-\frac{1}{e} \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} & \begin{vmatrix} z-\frac{1}{e} & 0 \\ 0 & z \end{vmatrix} & -\begin{vmatrix} z-\frac{1}{e} & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ z-\frac{1}{e} & \frac{-1}{100}\left(1-\frac{1}{\sqrt{e}}\right) \end{vmatrix} & -\begin{vmatrix} z-\frac{1}{e} & 0 \\ 0 & \frac{-1}{100}\left(1-\frac{1}{\sqrt{e}}\right) \end{vmatrix} & \begin{vmatrix} \ \end{vmatrix} \end{bmatrix}}{\det(zI - A_{new})}\begin{bmatrix} 0 \\ \frac{1}{100}\left(\frac{1}{\sqrt{e}} - \frac{1}{e}\right) \\ 1 \end{bmatrix}$$

(4)

$$[zI - A_{new}]^{-1} = \frac{\begin{bmatrix} z(z-\frac{1}{e}) & -0 & 0 \\ -0 & z(z-\frac{1}{e}) & -0 \\ 0 & -\left[\left[z-\frac{1}{e}\right]\left(-\frac{1}{100}\left(1-\frac{1}{\sqrt{e}}\right)\right)\right] & \left[z-\frac{1}{e}\right]^2 \end{bmatrix}^T}{\left(z-\frac{1}{e}\right)^2 z}$$

$$= \begin{bmatrix} \frac{1}{z-\frac{1}{e}} & 0 & 0 \\ 0 & \frac{1}{z-\frac{1}{e}} & \frac{\frac{1}{100}\left(1-\frac{1}{\sqrt{e}}\right)}{z\left(z-\frac{1}{e}\right)} \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

$$H(z) = [0\ 1\ 0] \begin{bmatrix} \frac{1}{z-\frac{1}{e}} & 0 & 0 \\ 0 & \frac{1}{z-\frac{1}{e}} & \frac{\frac{1}{100}\left(1-\frac{1}{\sqrt{e}}\right)}{z\left(z-\frac{1}{e}\right)} \\ 0 & 0 & \frac{1}{z} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{100}\left[\frac{1}{\sqrt{e}}-\frac{1}{e}\right] \\ 1 \end{bmatrix}$$

$$= [0\ 1\ 0] \begin{bmatrix} 0 \\ \frac{1}{100}\left[\frac{1}{\sqrt{e}}-\frac{1}{e}\right]\frac{1}{z-\frac{1}{e}} + \frac{1}{100}\left[1-\frac{1}{\sqrt{e}}\right]\frac{1}{z\left[z-\frac{1}{e}\right]} \\ \frac{1}{z} \end{bmatrix}$$

(g)

$$H(s) = \frac{1}{100} \frac{\left[z\left(\frac{1}{\sqrt{e}} - \frac{1}{e}\right) + \left(1 - \frac{1}{\sqrt{e}}\right)\right]}{z\left(z - \frac{1}{e}\right)}$$

$$= \frac{1}{100} \frac{0.2387z + 0.3935}{z(z - .3679)}$$

(d) if a_k is just given at the sampling instant what will be the transfer function [no delayed time system]

$$A_d = e^{AT} \qquad b_d = \int_0^T e^{A\tau} b \, d\tau \qquad c_d^T = c^T = [0 \; 1] \qquad d_d = d$$

$$A_d = \begin{bmatrix} \frac{1}{e} & 0 \\ 0 & \frac{1}{e} \end{bmatrix} \qquad b_d = \int_0^T e^{A\tau} b \, d\tau = \int_0^T \begin{bmatrix} e^{-100\tau} & 0 \\ 0 & e^{-100\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^T \begin{bmatrix} 0 \\ e^{-100\tau} \end{bmatrix} d\tau = \begin{bmatrix} 0 \\ -\frac{e^{-100\tau}}{100} \end{bmatrix} \Bigg|_0^{T=0.01}$$

$$b_d = \begin{bmatrix} 0 \\ -\frac{1}{100}\left(e^{-1} - 1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{100}\left[1 - \frac{1}{e}\right] \end{bmatrix}$$

$$H(s) = c_d^T \left[zI - A_d\right]^{-1} b_d = [0 \; 1] \begin{bmatrix} z - \frac{1}{e} & 0 \\ 0 & z - \frac{1}{e} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{100}\left[1 - \frac{1}{e}\right] \end{bmatrix}$$

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$$H(z) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{c_d^T} \underbrace{\begin{bmatrix} \frac{1}{z-\frac{1}{e}} & 0 \\ 0 & \frac{1}{z-\frac{1}{e}} \end{bmatrix}}_{A_d} \underbrace{\begin{bmatrix} 0 \\ \frac{1}{100}\left(1-\frac{1}{e}\right) \end{bmatrix}}_{b_d}$$

$$= \begin{bmatrix} 0 & \frac{1}{z-\frac{1}{e}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{100}\left[1-\frac{1}{e}\right] \end{bmatrix} = \frac{1}{10} \frac{\left[1-\frac{1}{e}\right]}{z-\frac{1}{e}}$$

Check controllability

$$Q = \begin{bmatrix} b_d & A_d b_d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{100}\left[1-\frac{1}{e}\right] & \frac{1}{e}\frac{1}{100}\left[1-\frac{1}{e}\right] \end{bmatrix} \rightarrow \text{not completely controllable}$$

Check observability

$$O = \begin{bmatrix} c_d^T \\ c_d^T A_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{e} \end{bmatrix} \rightarrow \text{not completely observable}$$

(e) Check controllability of $\left[A_{new}, b_{new}, c_{new}^T\right]$ system

$$Q = \begin{bmatrix} b_{new} & A_{new} b_{new} & A_{new}^2 b_{new} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{10}\left[\frac{1}{\sqrt{e}}-\frac{1}{e}\right] & \frac{1}{e}\left[\frac{1}{100}\left(\frac{1}{\sqrt{e}}-\frac{1}{e}\right)\right]+\frac{1}{100}\left[1-\frac{1}{\sqrt{e}}\right] & \frac{1}{e}\left[\frac{1}{e}\left[\frac{1}{100}\left(\frac{1}{\sqrt{e}}-\frac{1}{e}\right)\right]+\frac{1}{100}\left(1+\frac{1}{\sqrt{e}}\right)\right] \\ 1 & 0 & 0 \end{bmatrix}$$

$\rightarrow \text{rank}(Q) = 2$ not completely controllable